CONTROL SYSTEMS
&
COMPONENT

FOR POLYTECHNIC 6TH SEMESTER STUDENTS

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M-TECH (E.I.S) ! B-TECH (E.T.C)

Topic 1 : Basic Control System Components

DEFINITIONS OF IMPORTANT TERMS

Plants

A plant is a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation.

Systems

A system is a combination of components that act together and perform a certain objective.

Disturbance

A disturbance is a signal that tends to adversely affect the value of the output of a system.

Feedback control

Feedback control refers to an operation that, in the presence of disturbances, tends to reduce the difference between the output of a system and some reference input and that does so on the basis of this difference.

Servo Systems

A servo system (or servomechanism) is a feedback control system in which the output is some mechanical position, velocity, or acceleration.

Automatic Regulating Systems

An automatic regulating system is a feedback control system in which the reference input or the desired output is either constant or slowly varying with time and in which the primary task is to maintain the actual output at the desired value in the presence of disturbances.

Closed–loop Control Systems

Feedback control systems are often referred to as *closed–loop control systems.*

Open–loop Control Systems

Those systems in which the output has no effect on the control action are called *open–loop control systems.*

Adaptive Control Systems

Adaptation implies the ability to self–adjust or self–modify in accordance with unpredictable changes in conditions of environment or structure. The control system having a candid ability of adaptation (that is, the control system itself detects changes in the plant parameters and makes necessary adjustments to the controller parameters in order to maintain an optimal performance) is called the *adaptive control system.*

CLASSIFICATION OF CONTROL SYSTEMS

Linear versus Nonlinear Control Systems

For linear systems, the principle of superposition applies. Those systems for which this principle does not apply are nonlinear systems. Most real life control system have non linear characteristics to some extent.

Time–invariant versus Time–varying Control Systems

A time invariant control system (constant coefficient control system) is one whose parameters do not vary with time. A time–varying control system is a system in which one or more parameters vary with time. The response depends on the time at which an input is applied.

Continuous–time versus Discrete–time Control Systems

In a continuous–time control system, all system variables are functions of a continuous time t. A discrete–time control system involves one or more variables that are known only at discrete instants of time.

Single–input, Single–output versus Multiple –input, Multiple–output Control Systems

A system may have one input and one output. Such a system is called a single–input, single–output control system. Some systems may have multiple inputs and multiple outputs.

Lumped–parameter versus Distributed–parameter Control Systems

Control systems that can be described by ordinary differential equations are lumped–parameter control systems, whereas distributed–parameter control systems are those that may be described by partial differential equations.

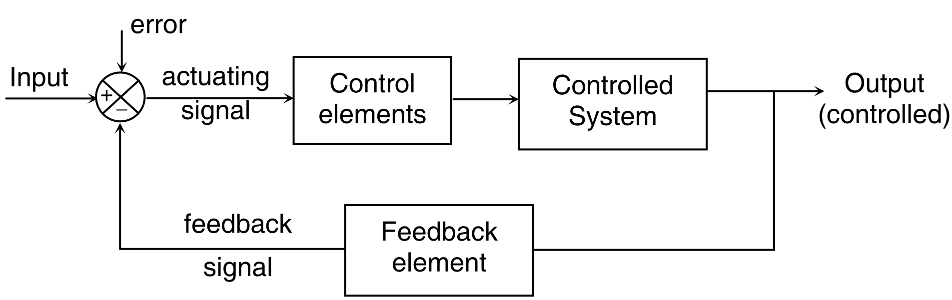
Deterministic versus Stochastic Control Systems

A control system is deterministic if the response to input is predictable and repeatable. If not, the control system is a stochastic control system.

INTRODUCTION TO CONTROL SYSTEMS

A closed loop control system consists of three basic elements : the feedback element, controller and controlled system.

The controller consists of error detector and control elements.



> The control element manipulates the actuating signal preferably to different power stage so as to fed to the controlled system.

Note :

The power stage in control elements is essential for the control signal to drive controlled system.

Control elements plays a vital role to get the desired output.

CONTROLLER COMPONENTS

They can be classified in three kinds
→ Sensors
→ Differencing and amplification
→ Actuators

1. Sensors
   - low power transducers
   - controlled variable
   - employed for position, velocity, measurement etc.

2. Differencing and amplification
   - to get error signal and amplification to suitable level
   - OPAMP is used for differencing input and feedback signals.
   - SCR is used for different power stages

3. Actuators
   - It is a device whose output is mechanical motion
   - It performs variety of tasks to manipulate the controlled process or plant. For example open/close a valve in a plant.

Actuators are classified as :

Actuators
↓
Pneumatic Actuators | Electric Actuators | Hydraulic Actuators

Electric actuators have inherent flexibility in electrical power transmission and have linear speed torque characteristics which is desired. Hence among all the actuators, electric actuators are widely used.

> For low speed and high torque applications, hydraulic actuators are used.

ELECTRIC ACTUATORS

Electric Actuators

- DC motors
- AC motors
- Stepper motors

In lower power ratings, these are called as *Servomotors.*

– DC motors are costlier than AC motors. This is because of additional cost of communications gear.

– DC motors have linearity of characteristics, higher stalled torque/ inertia ratio.

> Higher torque/ inertia ratio indicates dynamic response of the motor.

Electric actuators for stepped motion are known as *Stepper motors*.

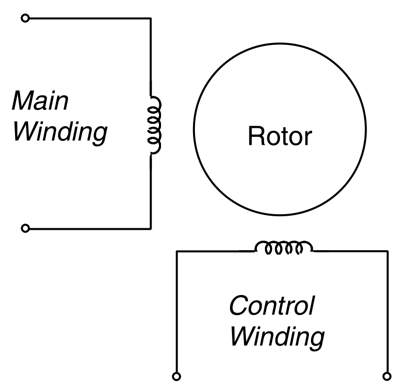
SERVOMOTORS (AC/DC)

Servomotors :

The commonly used power devices in electrical control systems are AC and DC servomotors. AC servomotors are ideally suited for Low Power application. They are rugged, light in weight and have no brush contacts.

AC Servomotor

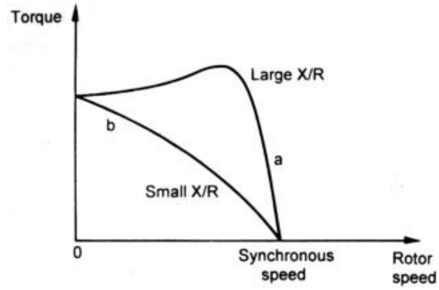
It is basically two phase induction motor



The stator windings are oriented 90° apart. Hence this results into magnetic field of constant magnitude rotating at synchronous speed.

> The direction of rotation depends upon phase relationship of voltages V_1 and V_2 Because of magnetic field, voltage is induced resulting into current in rotor. This current produces torque in the rotor.

Torque Speed Characteristics



where Rotor reactance : X
Rotor resistance : R

> $\frac{X}{R}$ ratio is generally high to obtain maximum starting torque, stable operation and to get linear torque–speed characteristics.

If symmetrical components are used, then the starting torque is proportional to E, (rms value of the sinusoidal voltage).

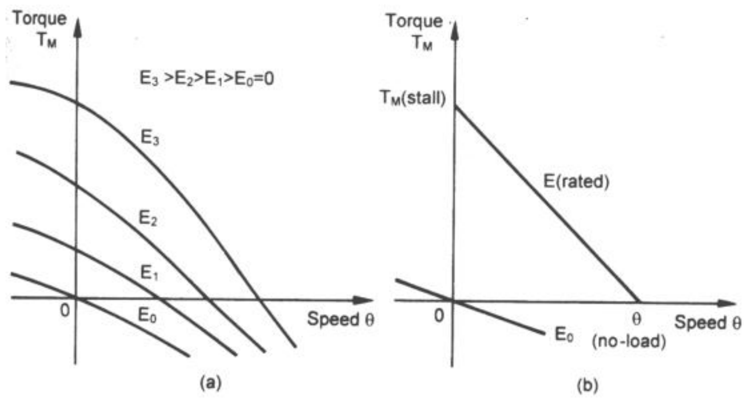


Fig. Servomotor characteristics

Note :

In low speed region, the curves are nearly linear and equidistant.

It reveals :

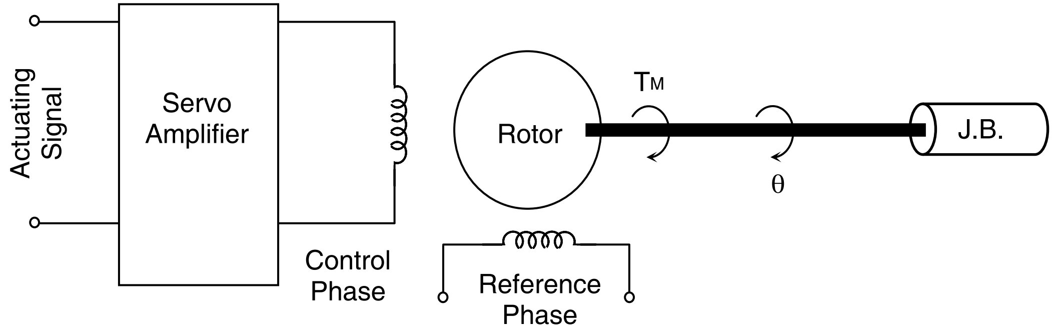
- Slope of torque speed characteristic reduces as control phase voltage decreases.
- Torque speed slope in low speed region is nearly one half the slope at rated voltage.

Approximations

1) The curves are approximated to linear characteristics. This approximation is valid as the motor rarely operates at high speeds.

2) The torque is proportional to applied voltage. Since the T–N curves are assumed linear and proportional to applied voltage, these curves are equally spaced.
The motor torque T_m is given by,

$$T_m = k_{tm} E + m \frac{d\theta}{dt}$$



where $E \rightarrow$ control voltage , $\frac{d\theta}{dt} \rightarrow$ speed

It's Laplace is

$$T_m(s) = k_{tm} E(s) - ms\,\theta(s)$$

The load has J and B components

$$\therefore \quad T_L = (Js^2 + Bs)\,\theta(s)$$

As Motor drives this load

$$T_m = T_L$$

$$k_{tm} E(s) - ms\,\theta(s) = (Js^2 + Bs)\,\theta(s)$$

To get $\frac{\theta(s)}{E(s)}$ is the objective.

$$\therefore \quad k_{tm} E(s) = (Js^2 + Bs + ms)\,\theta(s)$$

$$\therefore \quad \frac{\theta(s)}{E(s)} = \frac{k_{tm}}{(Js + B + m)s} = \frac{k_{tm}}{s(B+m)\left\{1 + \frac{Js}{B+m}\right\}}$$

Put $\quad k_m = \frac{k_{tm}}{(B+m)}$ and $\tau_m = \frac{J_m}{(B+m)}$

$$\therefore \quad \frac{\theta(s)}{E(s)} = \frac{k_m}{1 + s\,\tau_m}$$

The SFG is easily given from following rearrangement.

$$\frac{\theta(s)}{E(s)} = \frac{k_{t\,m}}{Js^2 + (B+m)\;\;s} = \frac{k_{tm} / Js^2}{1 - \left(\frac{m+B}{Js}\right)}$$

It has one forward path of gain $\frac{k_{tm}}{Js^2}$ and loop gain $\left(\frac{m+B}{Js}\right)$ shown in figure (a) below.

The term m contributes to negative slope. This improves friction and improves stability. It is called internal damping of 2 phase AC servomotor.

The characteristics are easily determined by two tests. See fig.(a) below.

1) Blocked Rotor Test $\rightarrow$ (Speed = 0)
2) No load test $\rightarrow$ (Load torque = 0)

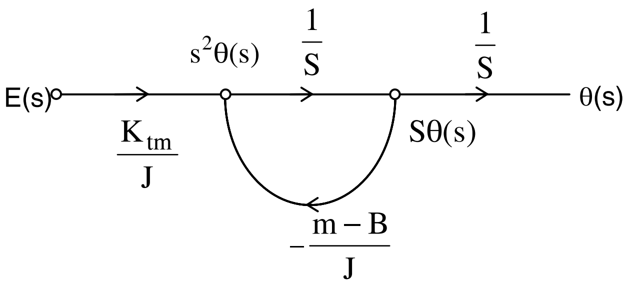


Fig. (a)

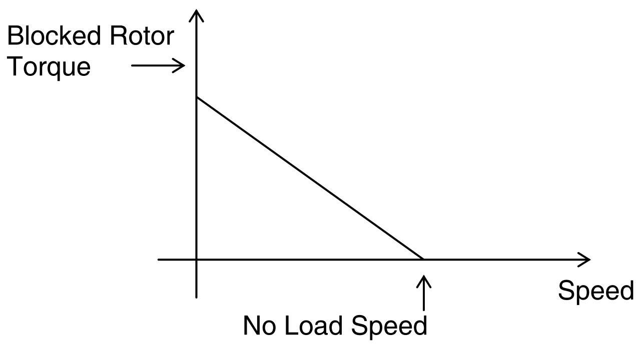


Fig. (b)

Fig. Permanent magnet dc motor (PMDC)

$$k_{tm} = \frac{\text{Blocked Rotor Torque at Rated voltage}}{\text{Rated Control voltage}}$$

$$m = -\frac{\text{Blocked rotor torque at rated voltage}}{\text{No load speed}}$$

DC Servomotors

These are constructed with permanent magnets which results into higher torque / inertia ratio and also higher operating frequency.

The DC servomotors can be again classified into 3 types

→ The dc servomotors in which magnetic field is produced by permanent magnet then magnetic flux is constant and it is called as *permanent magnetic DC Servomotor.*

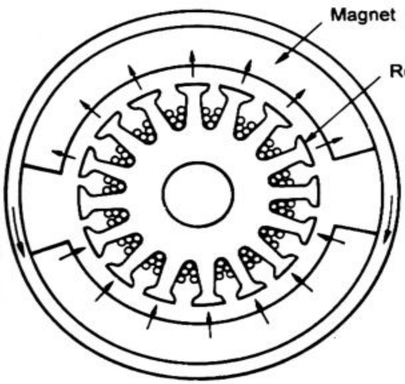
→ The dc servomotors in which output is controlled by armature current is called as *armature controlled DC servomotor.*

→ The dc servomotors in which armature current is maintained constant and field is controlled by armature current is called as *field controlled DC servomotor*.

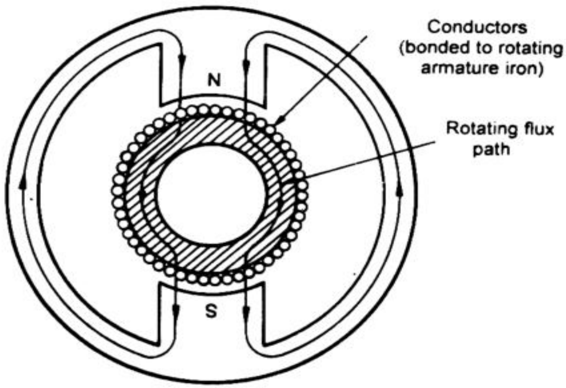
But providing a constant current source is more difficult than providing constant voltage source.

> The time constants of the field–controlled dc motor is large than that of armature controlled because of requirement of constant armature current.

Three types of construction employed in Permanent magnet DC servomotors are shown below



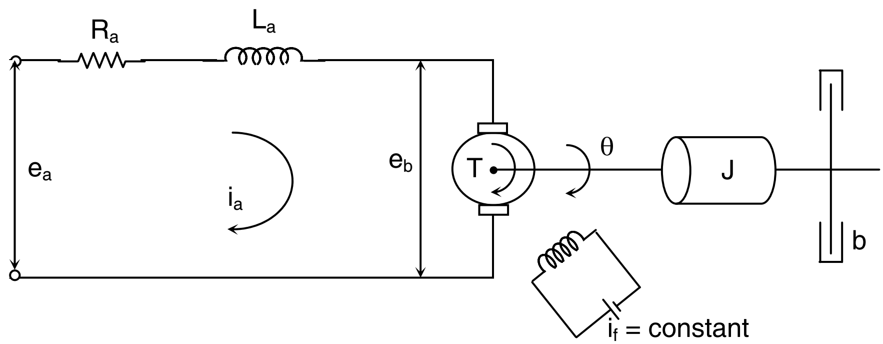
(a) Slotted armature type



(b) Surface wound armature type

Fig (a) : armature is slotted with DC winding placed inside these slots.
Fig (b) : winding placed on armature to reduce high inertia accounted in figure (a)
Fig (c) : winding placed on a non magnetic cylinder which rotates in annular space between PM stator and stationary rotor.

(a) Armature controlled dc servomotor:



R_a = armature resistance (Ω)
L_a = armature inductance (H)
i_a = armature current (A)
i_f = field current (A)
e_a = applied armature voltage (V)
e_b = back emf (V)
θ = angular displacement of motor shaft (radian)
T = Torque by motor (Nm)
J = Equivalent moment of inertia of the motor and load referred to motor shaft ($kg\ m^2$)
b = Equivalent viscous friction coefficient (Nm/rad/sec)

The torque is directly proportional to product of armature current and flux in air gap (ψ)
Also flux is directly proportional to field current.

$\therefore \quad \phi = K_1 i_f$ (By convention, ϕ is used for flux & ψ is used for flux linkages)

$$T = K_2 i_a \phi$$
$$= K_1 K_2 i_a i_f \phi$$

$\therefore \quad T = K i_a$ [$\because$ for constant field current, flux is constant]

When armature is rotating, voltage proportional to the product of flux and angular velocity is induced in armature. But flux is constant.

$$\therefore \quad e_b = K_4 \frac{d\theta}{dt} \qquad \text{....(1)}$$

e_b = back emf ($K_4 = K_b$: back emf constant)

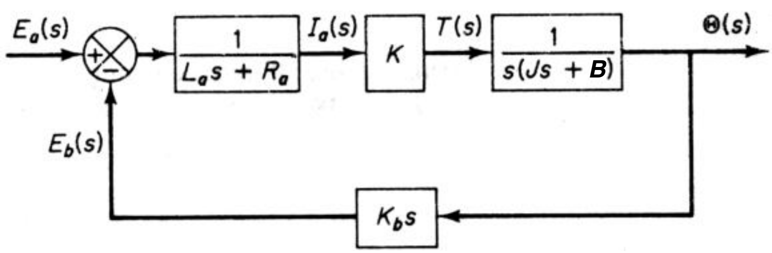
The differential equation for armature circuit is,

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \qquad \text{....(2)}$$

The torque equation can be given as

$$T = J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = K \cdot i_a \qquad \text{....(3)}$$

By taking the Laplace transform of above equations (1), (2), (3) then block diagram can be constructed as shown below.



It reveals :

- Armature controlled dc servomotor is a feedback system.
- Effect of back emf is the feedback system.
- Back emf increases with effective damping of system.

The transfer function is given as

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s\left[L_a J s^2 + (L_a B + R_a J)s + R_a B + K K_b\right]} \qquad \text{...(4)}$$

It reveals :

- $\frac{1}{s}$ term indicates system posses integrating property
- Time constant of motor is smaller for smaller R_a and smaller J.

If L_a is small and neglected then transfer function becomes

$$\frac{\theta(s)}{E_a(s)} = \frac{K_s}{s(T_s s+1)} \qquad \ldots(5)$$

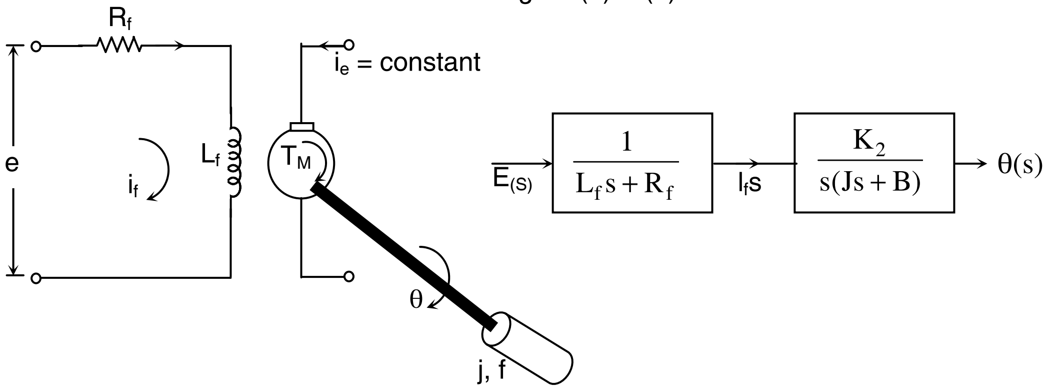
where $K_s = \dfrac{K}{R_a b + K K_b}$ = motor gain constant = $\dfrac{K}{R_a\left(B + K\,{K_b}/{R_a}\right)}$

$T_s = \dfrac{JR_a}{(R_a B + K K_b)}$ = motor time constant= $\dfrac{J}{\left(B + K_t\,{K_b}/{R_a}\right)}$

With small J, R_a reduced, the motor time constant approaches zero and motor acts as an ideal integrator.

(b) Field Controlled D.C. Motor

Field Controlled d.c. motor is shown in figure (a) & (b) below.



In this system,

R_f = Field winding resistance (ohms)

L_f = field winding inductance (henrys)

e = field control voltages (volts)

i_f = field current (amperes)

T_M = torque developed by motor (newton-m)

J = equivalent moment of inertia of motor and load referred to motor shaft (kg-m^2)

B = equivalent viscous friction coefficient of motor and load referred to motor shaft $\left(\dfrac{\text{newton}-\text{m}}{\text{rad / sec}}\right)$

θ = angular displacement of motor shaft (rad)

In the field controlled motor, the armature current is fed from a constant current source. Therefore, $T_m = k_1\, f_f\, i_f\, i_a = k_1'\, t_f$ where k_1' is a constant.

The equation for the field circuit is :

$$L_f \frac{di_f}{dt} + R_f i_f = e \qquad \ldots\ldots(6)$$

The torque equation is :

$$J \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T_M = k_r{'} \quad i_f \qquad \ldots\ldots(7)$$

Taking the Laplace transform of equations (6) & (7). Assuming zero initial conditions, we get

$$(L_f\, s + R_f)\, I_f\,(s) \;=\; E(s) \qquad \ldots\ldots(8)$$
$$(J\, s^2 + Bs)\,\theta(s) \;= T_M\,(s) = k_r'\; I\; f(s) \qquad \ldots\ldots(9)$$

From the above equations, the transfer function of the motor is obtained as :

$$\frac{\theta(s)}{E(s)} = \frac{K_r{'}}{s(L_f s + R_f) \qquad (Js + B)}$$
$$= \frac{K_m}{s \quad \left(\frac{\tau}{s+1}\left(\tau_{me} s + 1\right)\right)} \qquad \ldots\ldots(10)$$
$$= \frac{K_m}{s\left(1 + s\tau_f\right)\left(1 + s\tau_m\right)}$$

where K_m = $K_r' / R_f\, B$ = motor gain constant,

τ_f = L_f/R_f = time constant of field circuit and

τ_m = J/B = mechanical time constant

The block diagram of the field controlled d.c. motor obtained from eqns. (8) and (9) as given in fig.(b) above.

For small size motors field control is advantageous because only a low power servo amplifier is required while the armature current which is not large can be supplied from an inexpensive constant current amplifier. For large size motors it is on the whole cheaper to use armature control scheme. Further in armature controlled motor, back emf contributes additional damping over and above that of provided by load friction.

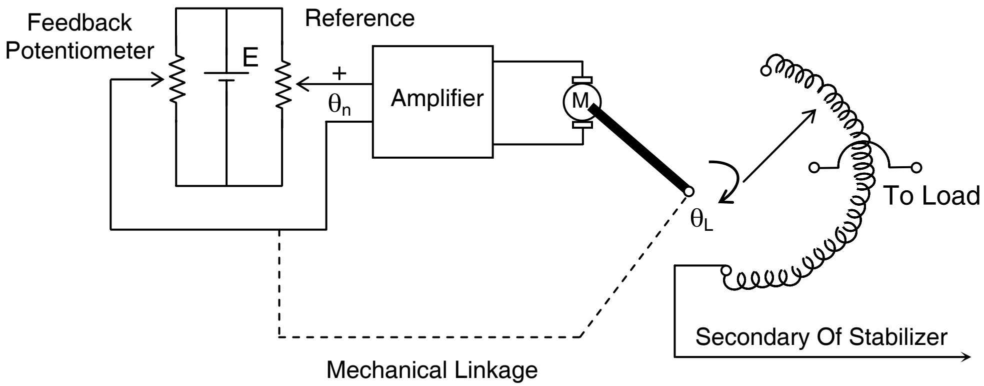
D.C. & A.C. Position Control :

In control systems DC signal refers to unmodulated signals. AC signal refers to modulated signals. These definition differ from our normal meaning of AC/DC.

Considering a servo voltage stabilizer.

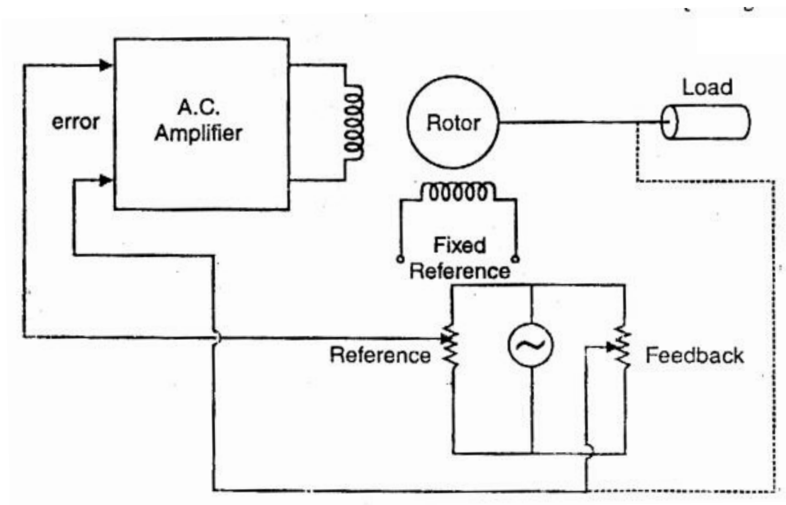
a) A typical DC position Control applied to servo stabilizer

The secondary winding variable tap position is driven by dc. servomotor. The feedback potentiometer wiper is similarly rotated by same θ as variable tapping. Suppose Reference voltage is higher than feedback voltage. This drives the motor in one direction. This causes wiper mounted on servomotor shaft to move, and output voltage increases. Simultaneously feedback voltage also rises as potentiometer turns moves in the direction towards reference value. The error decreases and will be zero when both these voltages are equal. At this moment, the motor stops rotating and further corrections are stopped.Thus the motor is driven by error voltage. The direction depends on the polarity of error.



b) A.C. Position Control :

In a.c. position control, the dc servomotor of above figure is replaced by A.C. servomotor. Again the error voltage drives the motor and polarity of error decides the CW or CCW sense of rotation. At zero error, control voltage is zero, so 2 phase ac servomotor cannot rotate. [see fig. below].



DC TACHOGENERATOR

Permanent magnet DC servomotor when coupled to a rotating shaft would generate voltage proportional to speed.

> Tachogenerator is also known as Tachometer. It is an another method for improving the performance of the servo system.

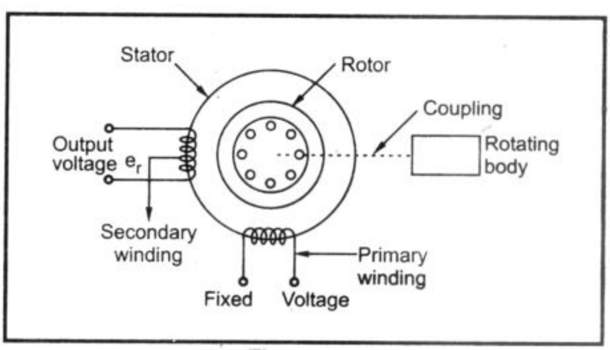
It can be viewed as a transducer, converting the velocity of shaft proportional to dc voltage.

- The DC Tachometer provides visual speed readout of a rotating shaft.
- Such tachometers are directly connected to a voltmeter which is calibrated in r.p.m.
- Permanent magnet tachometers are compact, efficient and reliable but have high inertia. To reduce inertia of rotors ironless rotors can be used.
- Permanent magnet units are compensated with temperature sensitive magnetic shunts that divert portion of pole flux according to temperature variation to maintain linear relationship between speed and the generated voltages.

Advantages

i) Generated voltages are free from undesirable waveforms and phase shifts
ii) No residual voltage is present at zero speed
iii) Possible to generate very high voltage gradients in small size.
iv) Can easily be compressed for temperature changes.
v) Can be used with high pass output filters to reduce servo velocity tags.

AC TACHOMETER



- For an A.C. tachometer, a sinusoidal voltage of rated value is applied to the primary winding which is also known as reference winding.

- The secondary winding is placed at a 90° mechanically apart in space from the primary winding.

- When rotor shaft is rotated, the magnitude of the sinusoidal output voltage e_T will be proportional to rotor speed. Thus, when the rotor shaft is stationary, the output voltage is zero. The phase of the output voltage is determined by direction of rotation.

The transfer function of an a.c. tachometer is,

$$e(t) = K\frac{d\theta}{dt}$$

i.e. $$\frac{E(s)}{\theta(s)} = Ks$$

where $E(s) \rightarrow$ Laplace Transform of output voltage

$\theta(s) \rightarrow$ Laplace Transform of the rotor position

$K \rightarrow$ constant

OR

$$\boxed{E_o = K_T \,.\, \omega}$$

Where K_T = Tachometer constant V/rpm

ω = Shaft speed in rad/sec.

Even though the output of an ac tachometer is an a.c. voltage, then also tachometer can be used in a d.c. servomechanism.

Because : If the output a.c. voltage is converted into a dc voltage by use of a demodulator.

> In servomechanism a.c. tachometer generators are used to provide the output rate damping.

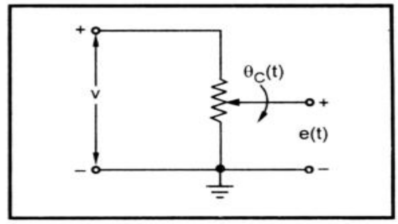
Advantages

i) A.C. Tachometer can be used as speed measuring devices.
ii) Also, A.C tachometer can be used as electro mechanical integrator in analog computers

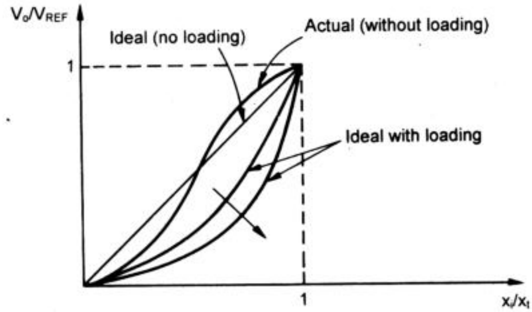
POTENTIOMETER

These are the devices used for measuring mechanical displacement. Potentiometer is a position electromechanical transducer that converts mechanical voltage into an electrical voltage. The input is in form of mechanical displacement either translatory linear or angular. A potentiometer is a simple voltage divider with three terminals. But two terminals are fixed and third is movable.

The voltage output across the movable terminal and reference is proportional to the displacement. But the linear relationship is affected by the magnitude of load.



Potentiometer Characteristics



When the housing of potentiometer is fixed at reference, the output voltage e(t) will be proportional to the shaft position $\theta_c(t)$ in case of a rotary motion then

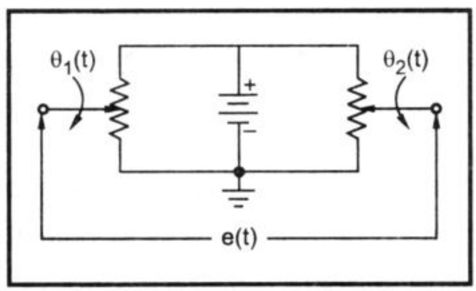
$$e(t) = k_s \theta_c(t)$$

where k_s is the proportionality constant.
For N turn potentiometer value of k_s is given by,

$$k_s = \frac{V}{2\pi N} \text{V / rad}$$

$$k_S = \frac{V}{\theta_{max}} \text{V / rad}$$

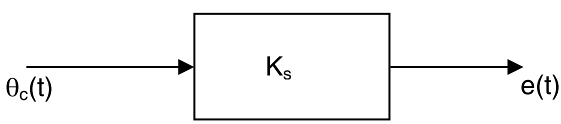
V = magnitude of reference voltage



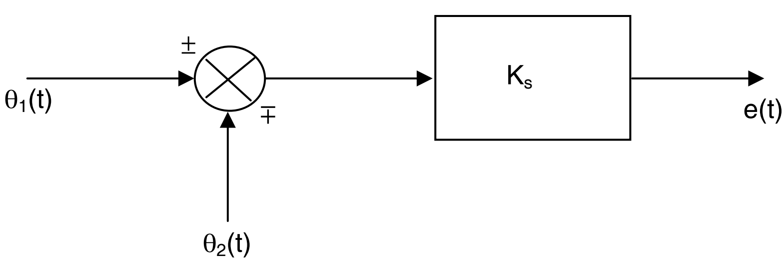
This type allows the comparison of 2 remotely located shaft positions. The output voltage is taken across the variable terminals of the 2 potentiometers.

$$\text{Output} \;\; e(t) = k_s\left[\theta_1(t) - \theta_2(t)\right]$$

Block diagram representation of the above two setups are shown below



Fig(1)



Fig(2)

Characteristics of Precision Potentiometer:

Performance of a precision potentiometer is specified by following characteristics

i) **Resolution**

It is the smallest incremental change that is possible in a potentiometers. It is the ratio of minimum change in output voltage to a total voltage applied to it.
For wire wound potentiometer it is per turn voltage.

$$\% \text{ Resolution} = \frac{\Delta V_o}{V_i} \times 100 = \frac{100}{\text{Number of turns}}$$

ΔV_o = Change in output voltage

V_i = Input voltage applied

> The range of resolution is between 0.5 to 0.002 %
>
> Potentiometer with uniformly spread resistance, resolution is infinite.

ii) **Linearity**

It is defined as the maximum deviation of actual curve from theoretical curve expressed as a percentage of applied voltage.

$$\% \text{ Linearity} = \frac{\Delta V_{max}}{V_i} \times 100 = \frac{\text{deviation in resistance}}{\text{actual resistance}} \times 100$$

> The range of linearity lies between 0.5 to 5.0 %

iii) **Life**
Life is defined as maximum number of cycles of operation in which none of electrical characteristics depart from normal values by more than 50%.

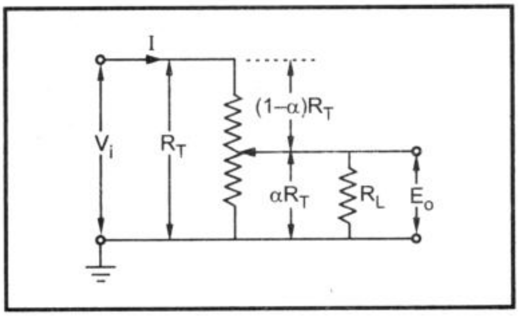
iv) **Noise**

Noise indicates presence of various voltages. Noise is due to ripple caused by vibration, stray capacitances, etc.

v) **Loading error**

As soon as the load is connected across the potentiometer, then the resistance of the potentiometer gets affected. This effect causes error in its output which is called as loading error.

Loading in Potentiometers



Let R_L be the resistance of the load connected to potentiometer and α be the setting ratio as shown in the figure.

The output voltage E_0 is given by

$$E_0 = \frac{\alpha V_i}{1 + \frac{\alpha(1-\alpha)R_T}{R_L}}$$

The loading error is,

$$\text{Error} = \alpha V_i - \frac{\alpha V_i}{\left[1 + \frac{\alpha(1-\alpha)R_T}{R_L}\right]}$$

$$\text{Error} = \left[\frac{\alpha^2(1-\alpha)}{\alpha(1-\alpha) + \frac{R_L}{R_T}}\right] V_i$$

SYNCHROS

An example of electromagnetic transducer that converts angular position of a shaft into electric signal is a synchros. It is also known as selsyn or autosyn.

Synchro Transmitter

In a synchro – transmitter, the construction is similar to alternator. Here rotor is dumb bell shaped and mounted with concentric coil. The stator is a 3 ϕ winding spaced 120° apart in space from each other. See fig. (a).

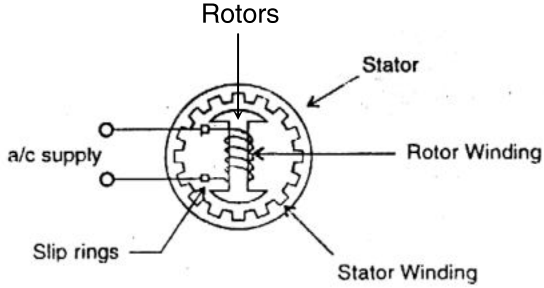


Fig. (a)

The rotor is given 1 phase a.c. supply. The voltage induced in each stator winding is proportional to cos θ as shown in figure (b). If reference voltage to rotor is :

$v_r = V \sin \omega t$

Then stator voltages w.r.t. neutral are :

$V_{1n} = kV \sin \omega t \cos (\theta + 120)$

$V_{2n} = kV \sin \omega t \cos (\theta)$

$V_{3n} = kV \sin \omega t \cos (\theta + 240)$ ......(1)

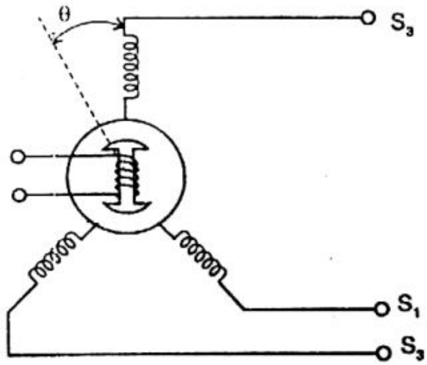


Fig (b)

Since the 3 terminals are accessible :

$V_{s1, s2} = V_{1n} - V_{2n} = kV \sin \omega t \cos (\theta + 120 - \cos (\theta))$

$V_{s1, s2} = \sqrt{3}\ k V \sin \omega t \sin (\theta + 240)$

$V_{s2, s3} = \sqrt{3}\ kV \sin \omega t \sin (\theta + 120)$ ......(2)

$V_{s3, s1} = \sqrt{3}\ kV \sin \omega t \sin (\theta)$.

At $\theta = 0$, $V_{s3, 1} = 0$ and this is electrical zero of transformer.

The synchro transmitter is thus like a single phase transformer with rotor as primary and 3 winding secondary. The voltages are in time phase, but with different magnitudes depending on θ. Thus a rotor position θ reflects as voltage in stator side.

Synchro Transmitter Receiver

When both rotors are excited by same input, both secondary also are tied, as shown in fig. below, then the rotor angular position will be followed by other rotor as per the first rotor.

Here Master rotor is rotated by θ_1 and slave also will move by similar amount θ_1.

– here rotor applied voltage, rotor angle and secondary voltages are the parameters.

→ In a control transformer, first two are fixed in transmitter. The rotor secondary (stator) voltage and rotor angle are fixed in receiver. Thus rotor voltage will change.

→ In transmitter receiver rotor voltages and stator voltages are same in both. Hence rotor position will follow.

It is used in detecting wind direction and other applications where change of angular position is seen.

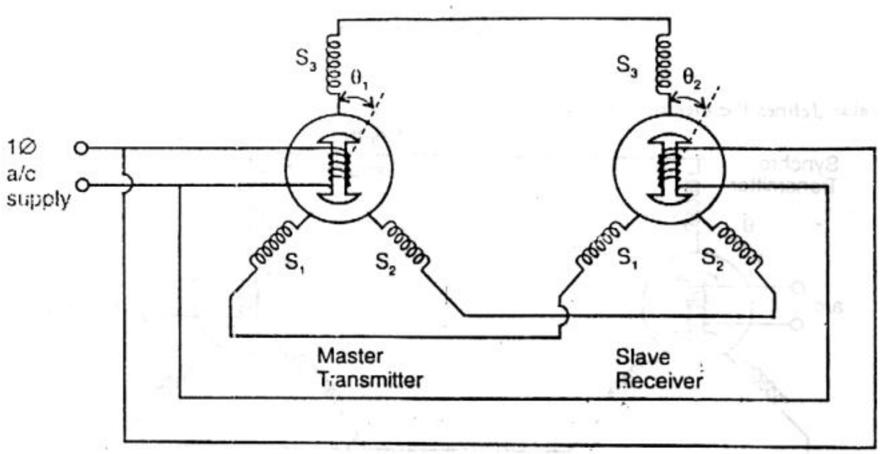


Fig.: Synchro Transmitter

STEPPER MOTOR

A stepper motor is widely used in computer peripherals like printers, tape drives etc. It is basically an electromagnetic device. It actuates movements, linear or angular, for a train of input pulses. One pulse gives one unit movement. The number of pulses gives the required number of movements desired.

Constructionally, there are two types of stepper motors.

1) Variable Reluctance Motor
2) Permanent Magnet Motor

The variable reluctance motor is taken up for discussion. It consists of one or several stacks of stators and rotors. The rotors are mounted on a single shaft. The stators have common frame. Fig. (a) and fig. (b) gives the two views of the stator–rotor arrangements in a stepper motor.

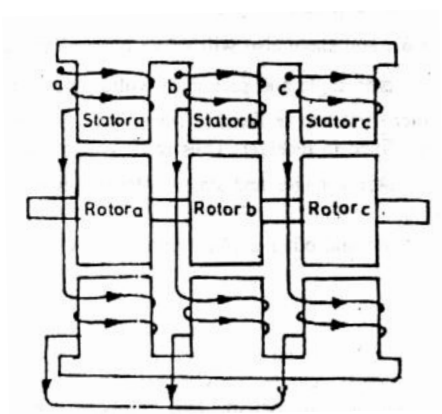


Fig. (a)
Longitudinal Cross Sectional view of 3 stack variable reluctance stepper.

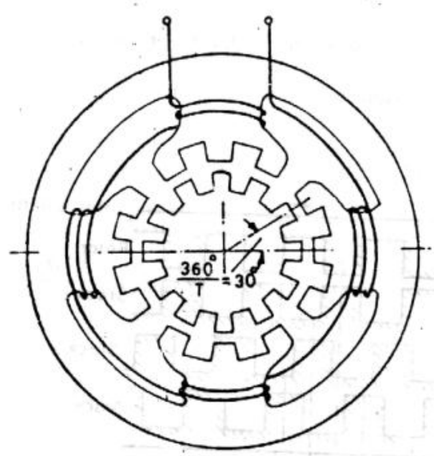


Fig. (b)
End view of stator & rotor of a multistack variable reluctance stepper motor.

The stator and rotor are of same size. The train of input pulses excite the stator. The rotor is not excited. As shown in figure, when the stator is excited, the rotor is pulled to a minimum reluctance position. The position occurs when the stator or rotor are aligned.

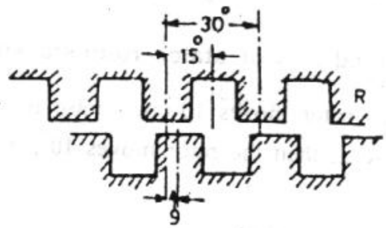


Fig. : Developed view of teeth of pair of stator – rotor

The position for $\theta = 0°$ is a stable position. The static torque acting on the rotor is a function of θ. The stators and rotors are aligned at $\theta = 0°$ and $\theta = \dfrac{180°}{T}$ where T is the number of rotor teeth. The latter value of θ is unstable and a slight deviation aligns the rotor at $\theta = 0°$. Any deviation for $\theta = 0°$, brings it back to the same position. Hence this position is a stable position. Fig. refers to the static torque angle curve.

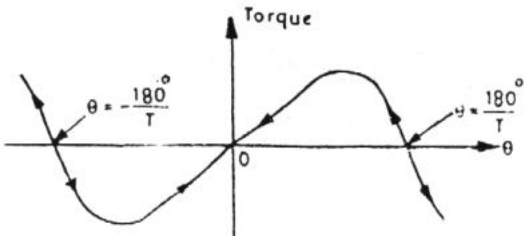


Fig. Static torque angle curve of stepper motor

When there are multiple stacks of a stator for a rotor with all teeth aligned, each stator has an angular displacement given by $\alpha = \dfrac{360°}{nT}$. Here n = number of stacks.

If T = 12 and n = 3, $\alpha = \dfrac{360°}{3.12}$ Fig. below gives this diagram.

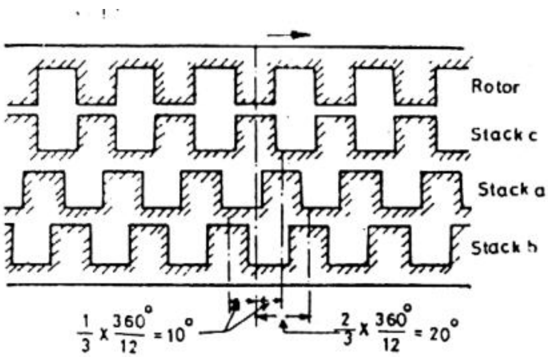


Fig. Developed view of stator–rotor stacks

If phase sequence is a-b-c-a-b-c … then rotor moves 10° per pulse for the case considered. If phase sequence is changed to b-a-c-b-a-c … then the rotor moves 10°, in opposite direction for each pulse.

Use of Stepper Motor in Control System :

Stepper Motors are used in open loop or closed loop mode.

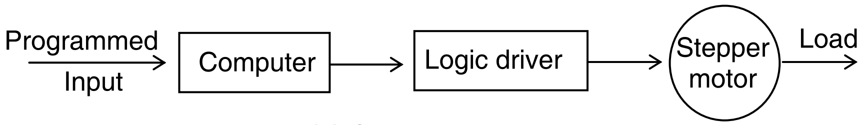


Fig. (a) Open loop mode

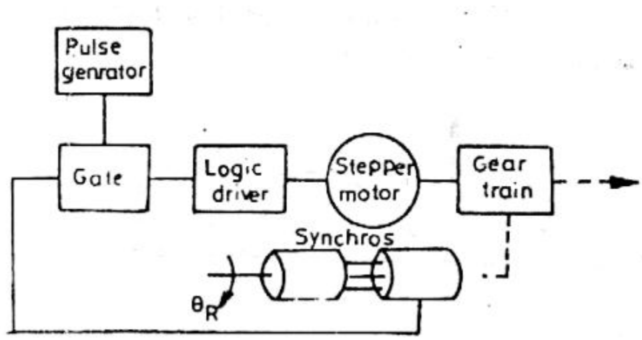


Fig. : (b) Closed loop mode

In open loop or closed loop, this is one device that gives same accuracy. This is because the angular displacement exactly equals the number of pulses given. So no feedback is normally needed. Figure (a) shows the open loop mode and fig. (b) shows the closed loop mode.

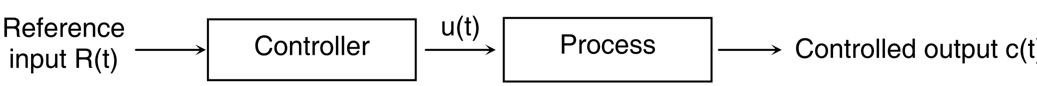
TYPES OF CONTROL SYSTEM

The control system may be classified into two types depending upon whether the controlled variable i.e., output affects the reference variable i.e., input or not.

The control systems are classified into two types :
1) Open loop control system
2) Closed loop control system

Open Loop Control System

A system in which the control action is totally independent of the output of the system is called as open loop system.



u(t) → actuating

→ Reference input R(t) is applied to the controller which generates the actuating signal u(t) required to control the process which is to be controlled. Process is giving out the necessary desired controlled output c(t).

Advantages of open loop system

1) They are simple in construction and design.
2) They are economical.
3) Easy for maintenance.
4) Not much problems of stability.
5) Convenient to use when output is difficult to measure.

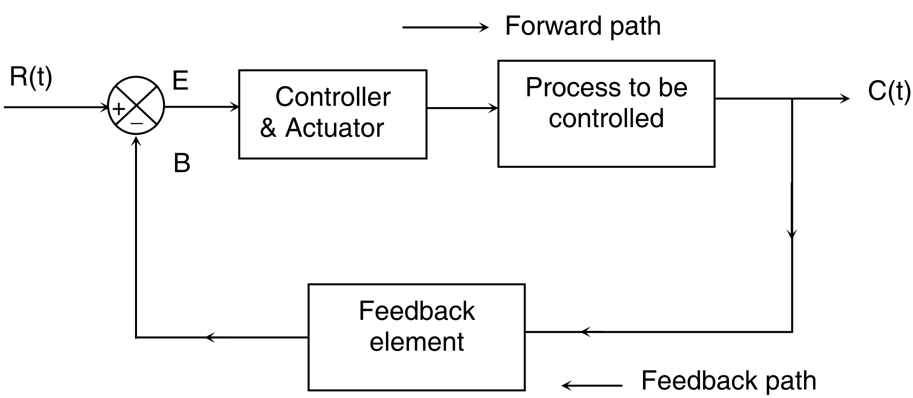
Disadvantages of open loop system

1) Inaccurate and unreliable because accuracy is dependent on accuracy of calibration.
2) Inaccurate results are obtained with parameter variations, internal disturbances.
3) To maintain quality and accuracy recalibration of the controller is necessary from time to time.

Closed Loop Control System

A system in which the controlling action is some how dependent on the output is called closed loop_control system. Such system uses a feedback.

A part of the output is feedback or connected to the input. i.e., feedback is that property of the system which permits the output to be compared with the reference input so that appropriate controlling action can be decided.



R(t) – Reference input

C(t) – Controlled output

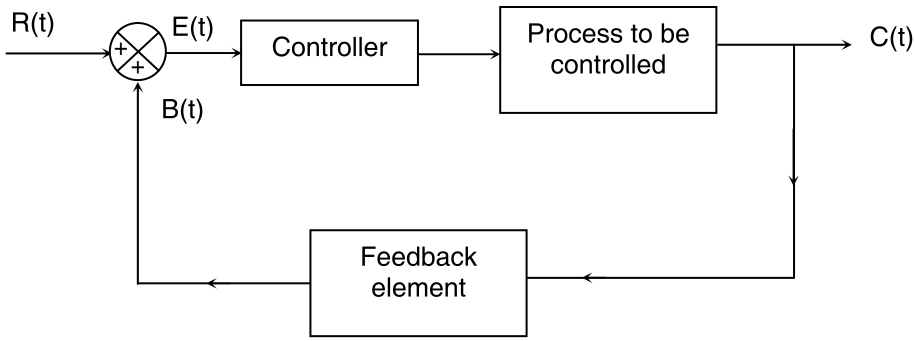
B – Feedback signal

E – Error signal.

There are two types of feedbacks.

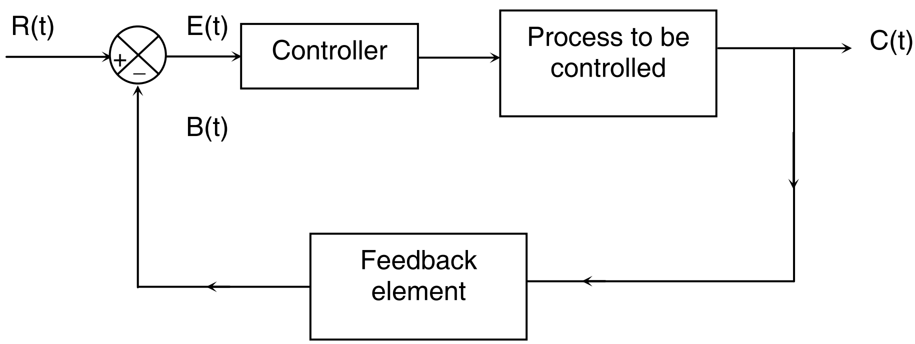
Positive Feedback (Regenerative feedback) :

When output is connected to input with + sign, then it is called as positive feedback.



Negative Feedback (Degenerative feedback):

When output is connected to input with – sign, then it is called negative feedback.



Advantages of closed loop system

1) Accuracy is very high as any error arising is corrected.
2) It senses changes in output due to environmental or parametric changes or internal disturbances.
3) Reduces effect of non–linearity.
4) Increases Bandwidth.

Disadvantage of closed loop system

1) Complicated in design.
2) Maintenance is costlier.
3) System may become unstable.

Effect of Feedback

When feedback is given the error between system input and output is reduced. However improvement of error is not only advantage. The effects of feedback are

1) Gain is reduced by a factor $\frac{G}{1 \pm GH}$.
2) Reduction of parameter variation by a factor $1 \pm GH$.
3) Improvement in sensitivity.
4) Stability may be affected.
5) Linearity of system improves
6) System Bandwidth increases

Difference between Open and Closed Loop Control System

Open loop		Closed loop	
1	Any change in output has no effect on the input. i.e., feedback does not exist	1	Changes in output, affects the input which is possible by use of feedback.
2	Output is difficult to measure	2	Output measurement is necessary
3	Feedback element is absent	3	Feedback element is present
4	Error detector is absent	4	Error detector is necessary
5	It is inaccurate and unreliable	5	Highly accurate and reliable
6	Highly sensitive to the disturbance	6	Less sensitive to the disturbances
7	Highly sensitive to the environmental changes	7	Less sensitive to the environmental changes
8	Simple in construction and cheap	8	Complicated to design and hence costly
9	System operation degenerates if the non-linearities present	9	System operates better than open loop system if the non-linearities present

LIST OF FORMULAE

- The motor torque T_m is given by,

$$T_m = k_{tm} E + m \frac{d\theta}{dt} \text{ where } E \rightarrow \text{control}, \frac{d\theta}{dt} \rightarrow \text{speed}$$

- % Resolution = $\frac{\Delta V_o}{V_i} \times 100$

  ΔV_o = Change in output voltage

  V_i = Input voltage applied

- % Linearity = $\frac{\Delta V_{max}}{V_i} \times 100$

- In Potentiometer, the loading error is,

$$\text{Error} = \alpha V_i - \frac{\alpha V_i}{\left[1 + \frac{\alpha(1-\alpha) R_T}{R_L}\right]}$$

$$\text{Error} = \left[\frac{\alpha^2 (1-\alpha)}{\alpha(1-\alpha) + \frac{R_L}{R_T}}\right] V_i$$

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- The control element manipulates the actuating signal preferably to different power stage so as to drive or feed to the controlled system.
- Control elements plays a vital role to get the desired output.
- For low speed and high torque applications, hydraulic actuators are used.
- Electric actuators have inherent flexibility in electrical power transmission and have linear speed torque characteristics which is desired.
- Higher torque/ inertia ratio indicates better dynamic response of the motor.
- In Servomotors the direction of rotation depends upon phase relationship of voltages V_1 and V_2 of the main winding and the control winding.
- In the AC servomotors, if symmetrical components are used, then the starting torque is proportional to E, (rms value of the sinusoidal voltage).
- The time constants of the field–controlled dc motor is large than that of armature controlled because of high inductance of field winding.

- An example of electromagnetic transducer that converts angular position of a shaft into electric signal is a synchros. It is also known as selsyn or autosyn.

- *Advantages of open loop system*

  1) They are simple in construction and design.
  2) They are economical.
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  3) System may become unstable.

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Topic 2 : Transfer Function, Block Diagram Reduction & Signal Flow Graph

TRANSFER FUNCTION

The relationship between input and output of a system is given by the transfer function.
For a linear time–invariant system the response is separated into two parts : the forced response and free response. The forced response depends upon the initial values of input and the free response depends only on the initial conditions on the output.

The transfer function P(s) of a continuous system is defined as

$$P(s) = \sum_{i=0}^{m} b_i s^i \Big/ \sum_{i=0}^{m} a_i s^i$$

$$= \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots\ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots\ldots\ldots + a_0}$$

The denominator is called the characteristic polynomial.

The transform of the response may be rewritten as

$Y(s) = P(s).U(s) +$ (terms due to all initial values)

If all the initial conditions are assumed zero then

$Y(s) = P(s)\ U(s)$

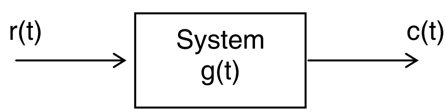
And the output as a function of time y(t) is simply

$$L^{-1}\left[Y(s)\right] = L^{-1}\left[P(s) \cdot U(s)\right] = y(t)$$

Definition

The transfer function is defined as the ratio of Laplace transform of output to Laplace transform of input under assumption that all initial conditions are zero.

For example :



r(t) $\rightarrow$ input
L r(t) = R(s)
c(t) $\rightarrow$ output
Lc(t) = C(s)
g(t) $\rightarrow$ system function
Lg(t) = G(s)

$\therefore$ Transfer function G(s) :

$$G(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}$$

$$\boxed{G(s) = \frac{C(s)}{R(s)}}$$

Properties

1. It is the Laplace transform of its impulse response $y_\delta(t), \quad t \geq 0$

2. The system transfer function can be determined from the system differential equation by taking the Laplace transform and ignoring all terms arising from initial values.

3. The system differential equation can be obtained from the transfer function by replacing the s variable with d/dt.

4. The stability of a time–invariant line system can be determined from the characteristic equation. Consequently, for continuous systems, if all the roots of the denominator have negative real parts, the system is stable.

5. The roots of the denominator are system poles and roots of the numerator are system zeros. The system transfer function can then be specified to within a constant by specifying the system poles and zeros. This constant k, is system 'gain factor'.

6. Transfer, function does not contain any information about the physical structures $\therefore$ system with different physical structure can have same transfer function.

7. Transfer function is the property of the system and does not depend upon the type of input.

Example :

Given P(s) = (2s + 1)/ (s^2 + s + 1)
The system differential equation is

Solution :

Replace all 's' by 'D'

$$P(s) = \frac{2s+1}{s^2+s+2}$$

$$y \quad = \left[\frac{2D+1}{D^2+D+1}\right]u$$

$$= D^2y + Dy + y$$

$$= 2Du + u$$

$$= \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 2\frac{du}{dt} + u$$

CONTINUOUS SYSTEM TIME RESONSE

The Laplace transform of the response of a continuous system to a specific input is given by

$$Y(s) = P(s)\,U(s) \qquad \text{when all initial conditions are zero.}$$

The inverse Laplace transform y(t) = L^{-1}[P(s) U(s)] is then the time response and y(t) may be determined by finding the poles of P(s) U(s) and evaluating the residues at these poles.

Advantages of Transfer Function

1) It is a mathematical model and gives the gain of given system.
2) As it uses a Laplace approach, it converts integro–differential time domain equation to simple algebraic equation.
3) Once transfer function is known, any output for any given input can be calculated.
4) It helps in determining the important information about the system i.e., poles, zeros, characteristics equations.
5) It helps in the stability analysis of the system.

Disadvantages of Transfer Function

1) Transfer function is valid only for linear time invariant system.
2) Effects arising due to initial conditions are totally neglected. Hence initial conditions loose their importance.

POLES AND ZEROS OF A TRANSFER FUNCTION

The transfer function is given by :

$$G(s) = \frac{C(s)}{R(s)}. \quad \text{Both C(s) and R(s) are polynomials in s}$$

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots\ldots. + a_n}$$

$$G(s) = \frac{k(s-b_1)(s-b_2)(s-b_3)\ldots\ldots(s-b_m)}{(s-a_1)(s-a_2)(s-a_3)\ldots\ldots(s-a_n)}$$ where k – system gain factor.

Then $b_1, b_2, \ldots b_m$ are called system zeros.

and $a_1, a_2, \ldots a_n$ are called as system poles.

For e.g., : $$G(s) = \frac{3(s+3)(s+1.5)^3}{(s+5)(s+7)^2} \qquad \ldots\ldots(1)$$

Poles

The value of s for which the system magnitude | G(s) | becomes infinity are called poles of G(s). When pole values are not repeated, such poles are called as simple poles. If repeated such poles are called multiple poles of order equal to the number of times they are repeated.

For example in equation (1) : poles are at s = –5 and s = –7.
The pole at s = –5 is simple pole and the pole at s = –7 is multiple pole of 2^{nd} order multiplicity.

Zeros

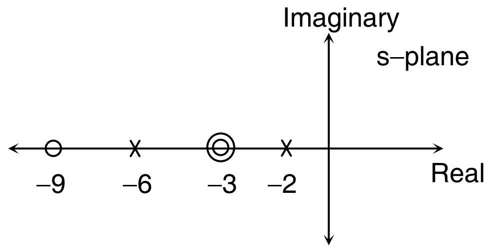
The value of s for which the system magnitude | G(s) | becomes zero are called zeros of transfer function G(s). When they are non repeated, they are called simple zero, otherwise they are called multiple zeros.

For example in equation (1), zeros are at s = –3 and s = –1.5.
The zero at s = –3 is simple zero whereas the zero at s = –1.5 is repeated of order three.

Representation of Pole and Zeros on S–plane

Zeros are represented by **o**
Poles are represented by **x**

$$G(s) = \frac{(s+3)^2(s+9)}{(s+6)(s+2)^2}.$$



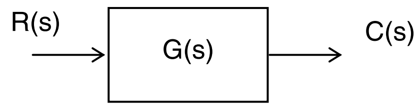
Characteristic Equation

The denominator polynomial of the closed loop transfer function of a closed loop system is called as characteristics equation and is given by

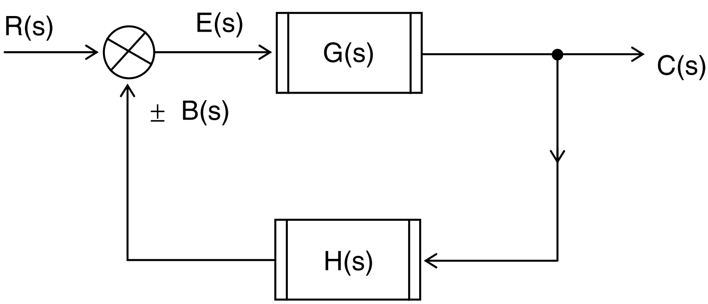
$$1 + G(s)\,H(s) = 0$$

Transfer Function of open loop control system

$$\boxed{\text{T.F.} = G(s) = \frac{C(s)}{R(s)}}$$



Transfer function of closed loop control system



where	R(s)	$\rightarrow$	Laplace transfer of ref. input R(t).
	C(s)	$\rightarrow$	Laplace transfer of con. o/p. C(t)
	E(s)	$\rightarrow$	Laplace transfer of error signal e(t).
	B(s)	$\rightarrow$	Laplace transfer of feedback signal b(t)
	G(s)	$\rightarrow$	Forward path transfer function
	H(s)	$\rightarrow$	Feedback path transfer function.

For above diagram :

$$E(s) = R(s) \pm B(s) \qquad \ldots\ldots(1)$$

$$H(s) = \frac{B(s)}{C(s)}$$

$$B(s) = H(s)\,C(s) \qquad \ldots\ldots(2)$$

$$G(s) = \frac{C(s)}{E(s)}$$

$$E(s) = \frac{C(s)}{G(s)} \qquad \ldots\ldots(3)$$

Substitute equation (2) in (1)

$$E(s) = R(s) \pm H(s)\,C(s) \qquad \ldots\ldots(4)$$

Substitute (3) in (4) :

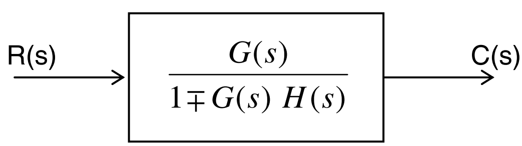
$$\frac{C(s)}{G(s)} = R(s) \pm C(s)\,H(s)$$

$$\therefore \quad C(s) = R(s)\,G(s) \pm C(s)\,H(s)\,G(s)$$

$$C(s) \mp C(s)\,H(s)\,G(s) = R(s)\,G(s)$$

$$C(s)\,[1 \mp H(s)\,G(s)] = R(s)\,G(s)$$

$$\boxed{\text{T.F.} = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)\,H(s)}}$$



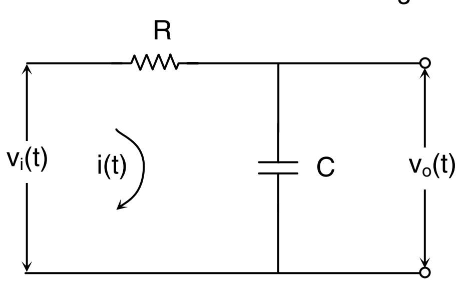
– sign → positive feedback.
+ sign → negative feedback.

Closed loop Transfer function

Transfer function of Electrical system

Problem 1 :

Find the transfer function of following electrical network.



$$v_i(t) = R_i\,(t) + \frac{1}{c}\int i(t)\,dt \qquad \ldots\ldots(1)$$

$$v_o(t) = \frac{1}{c}\int i(t)\,dt \qquad \ldots\ldots(2)$$

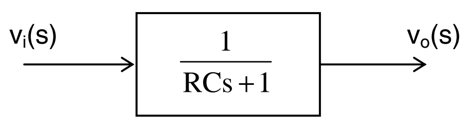
Take L. T. of equation (1) & (2). Assuming initial conditions zero.

$$v_i(s) = R\,I(s) + \frac{1}{sC}I(s) = \left(R + \frac{1}{sC}\right)I(s) = \frac{RsC+1}{sC}I(s) \qquad \text{......(3)}$$

$$v_o(s) = \frac{1}{sC}I(s) \qquad \text{......(4)}$$

From eqn. (3) & (4)

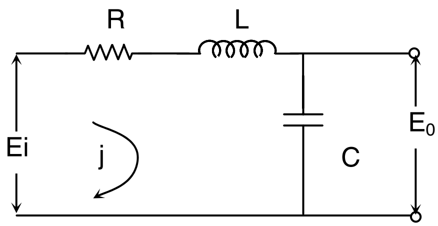
$$\text{T.F.} = \frac{v_o(s)}{v_i(s)} = \frac{\frac{1}{sC}I(s)}{\frac{(RCs+1)}{sC}\quad I(s)}$$



$$\text{T.F.} = \frac{v_o(s)}{v_i(s)} = \frac{1}{RCs+1}$$

Problem 2 :

Find out the T.F. of given network

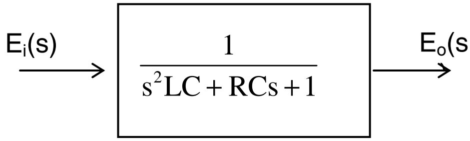


$$E_i = R\,i + L\,\frac{di}{dt} + \frac{1}{c}\int i\,dt \qquad \text{......(1)}$$

$$E_o = \frac{1}{c}\int i\,dt \qquad \text{......(2)}$$

Take L.T. of equation (1) & (2). Assuming initial condition zero.

$$E_i(s) = R\,I(s) + LS\;I(s) + \frac{1}{Cs}\;I(s)$$



$$E_i(s) = (R + LS + \frac{1}{sC})\;I(s) \qquad \text{......(3)}$$

$$E_o(s) = \frac{1}{sC}\;I(s) \qquad \text{......(4)}$$

From (3) & (4) :

$$\text{T.F.}= \frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs} \;.\; I(s)}{(R + Ls + \frac{1}{Cs})I(s)} = \frac{1}{RCs + s^2LC + 1}$$

$$\boxed{\text{T.F.}= \frac{E_o(s)}{E_i(s)} = \frac{1}{s^2LC + RCs + 1}.}$$

BLOCK DIAGRAM REDUCTION

In order to draw the block diagram of a practical system each element of practical system is represented by a block.

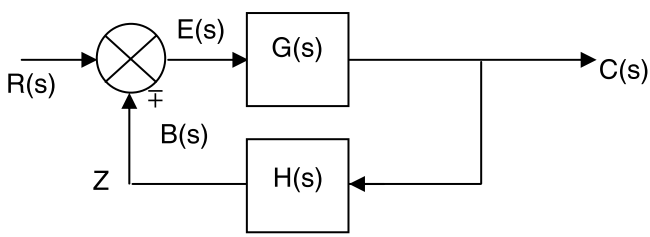
For a closed loop system, the function of comparing the different signals is indicated by the summing point while a point from which signal is taken for the feedback purpose is indicated by take off point in block diagrams.

A block diagram has following five basic elements associated with it.

1) Functional Blocks
2) Transfer functions of elements shown inside the functional blocks
3) Summing points
4) Take off points
5) Arrow

Transfer function of a Closed Loop System

(This representation is also called standard canonical form)



E(s) = R(s) $\mp$ B(s) …(1)

B(s) = C(s) H(s) …(2)

C(s) = E(s) G(s)

and substituting equation (2) in equation (1)

E(s) = R(s) $\mp$ C(s) H(s)

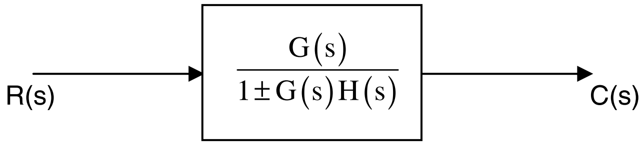
$$E(s) = \frac{C(s)}{G(s)}$$

C(s) = R(s) G(s) $\mp$ C(s) G(s) H(s)

$$C(s)\left[1 \pm G(s)H(s)\right] = R(s)G(s)$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

+ sign $\rightarrow$ negative feedback
– sign $\rightarrow$ positive feedback



Closed loop Transfer function

Rules for Block Diagram Reduction

Rule 1 : Associative Law

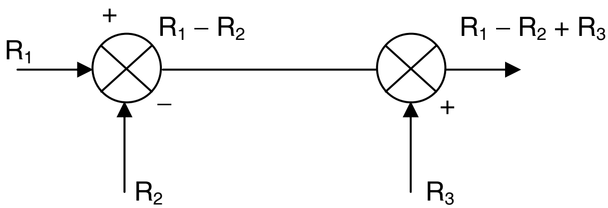


Fig. 1

Now even though we change the position of the two summing points, output remains same.

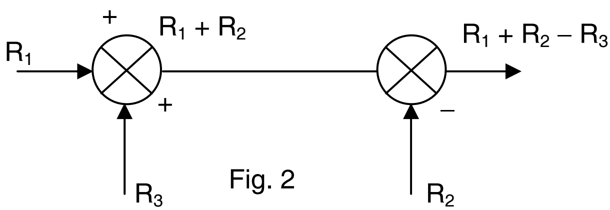


Fig. 2

Thus associative law holds good for summing points which are directly connected to each other.

Rule 2 :

For blocks in series

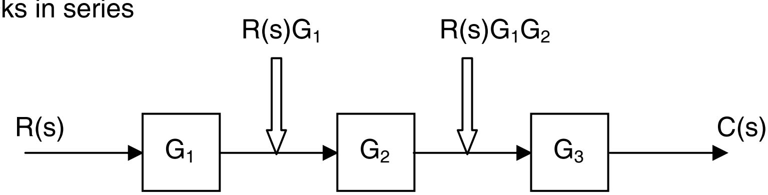


Fig. 1

$C(s) = R(s)[G_1 G_2 G_3]$

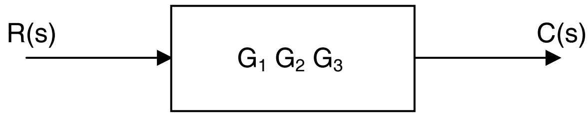


Fig 2

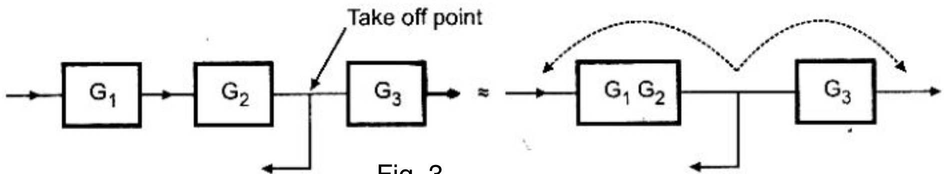


Fig. 3

Here G_1 and G_2 are in series and can be combined. But because of the take off point G_3 cannot be combined.

Rule 3 :

For blocks in parallel

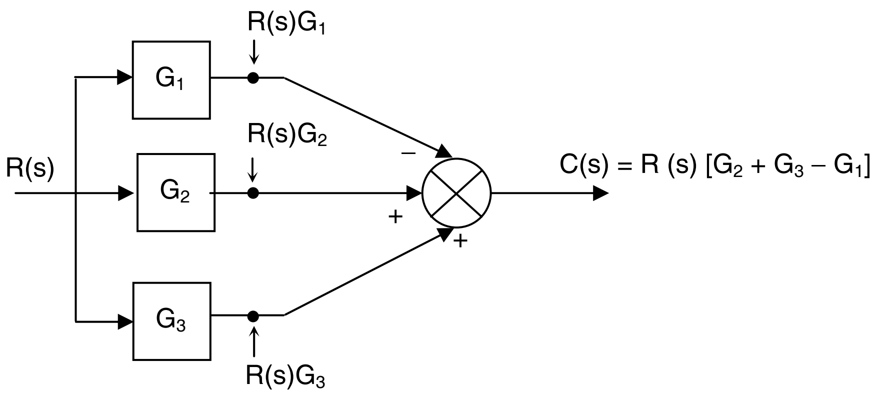


Fig. 1

C(s) = R(s) [$G_2 + G_3 - G_1$]

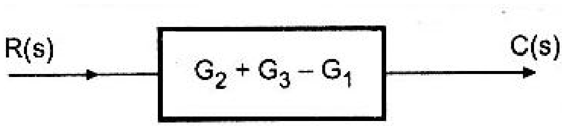


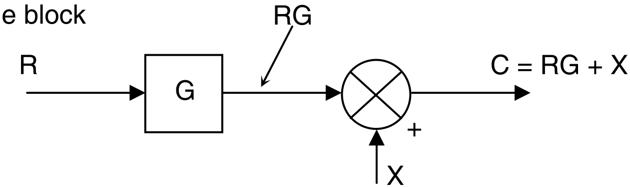
Fig. 2

> Blocks in parallel get added or subtracted depending on the sign of the summer.

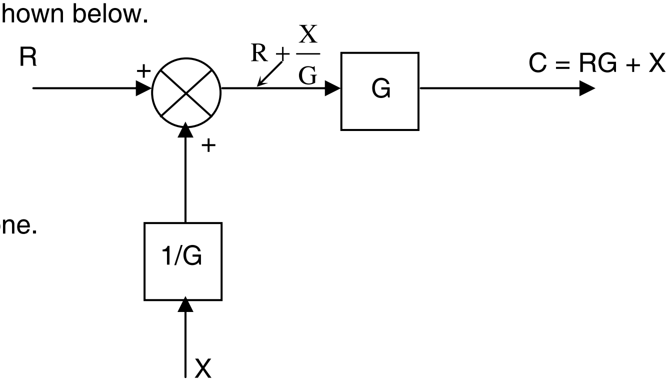
Rule 4 :

Shifting a summing point behind the block

C = RG + X



If we want to shift the summing point behind the block, and still get the same output certain modification is necessary as shown below.



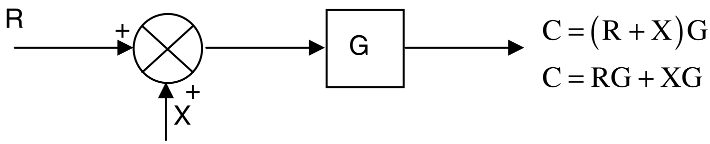
Now , $C = \left(R + \frac{X}{G}\right)G$

$\boxed{C = RG + X}$

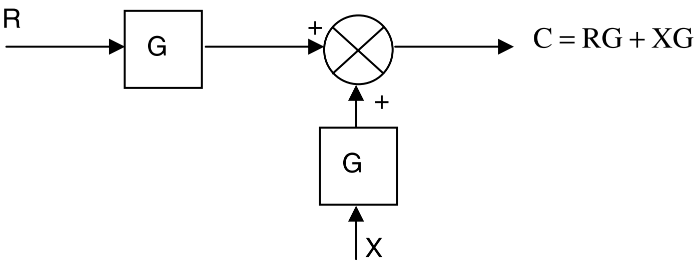
This output is same as the previous one.
Thus, the modification is correct.

Rule 5 :

Shifting a summing point beyond the block

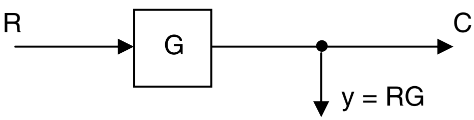


To shift the summing point beyond the block, we need to multiply the input X by G so that the same output is obtained.

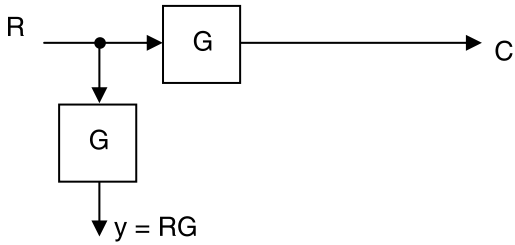


Rule 6 :

Shifting a take off point behind the block

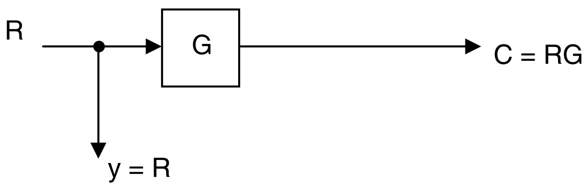


To shift the take off point behind the block, it is necessary to add a block G as shown below.

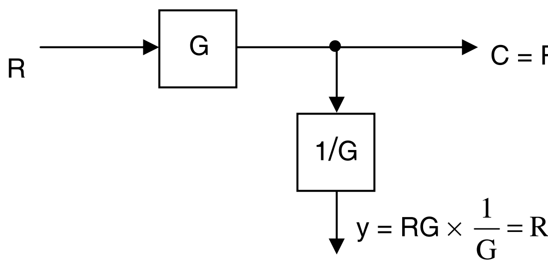


Rule 7 :

Shifting a take off point beyond the block :

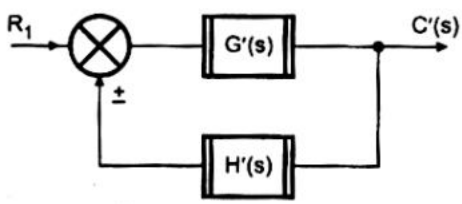


To shift a take of point beyond the block, the following modification is done.



Rule 8 :

Removing minor feedback loop :



≈

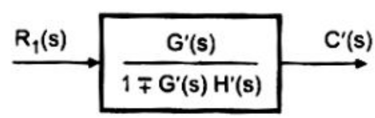
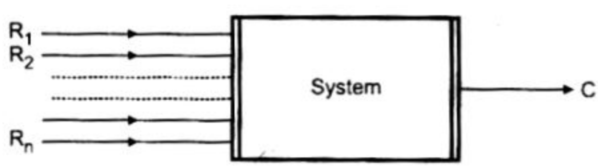


Fig. 1 Fig. 2

The minor loop shown in the figure 1 can be replaced directly by figure 2.

Rule 9 :

For multiple input system use superposition theorem (only if system is linear)



Consider only 1 input at a time treating all others as zero

Consider R_1, $R_2 = R_3 = \dots = R_n = 0$ and find C_1

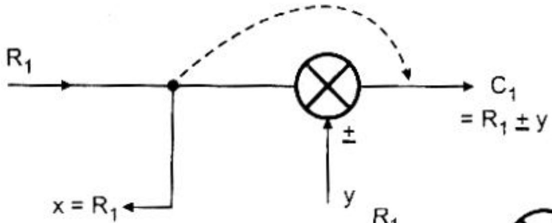
The consider R_2, $R_1 = R_3 = \dots = R_n = 0$ and find C_2

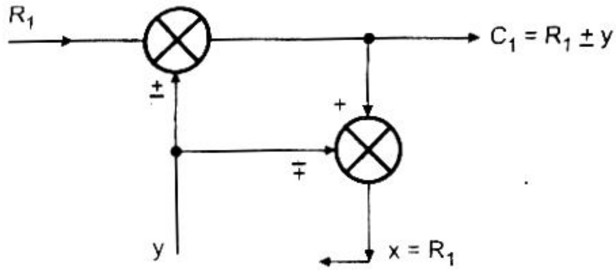
Total output $C = C_1 + C_2 + C_3 + \dots + C_n$

Critical Rules

Rule 10 :

Shifting take off point after a summing point :





If we want to shift the take off point after the summing point, 'y' input should be subtracted from the new take off point as shown in the figure below.

Rule 11 :

Shifting take off point before a summing point :

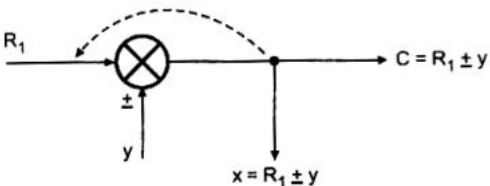


Fig. 1

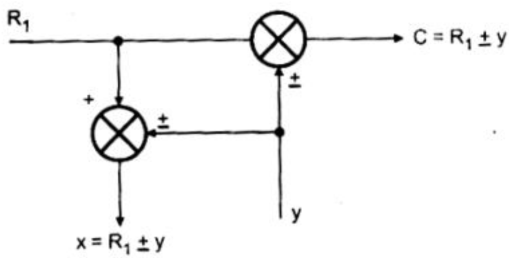


Fig. 2

Procedure to solve block diagram reduction Problems

Step 1 : Reduce the blocks connected in series.

Step 2 : Reduce the blocks connected in parallel.

Step 3 : Reduce the minor internal feedback loops.

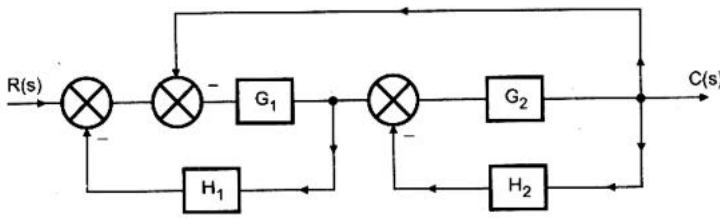
Step 4 : As far as possible try to shift take off point towards right and summing points to the left.

Step 5 : Repeat steps 1 to 4 till simple form is obtained, to get the final transfer function.

Step 6 : By using standard transfer function of simple closed loop system, obtain the closed loop

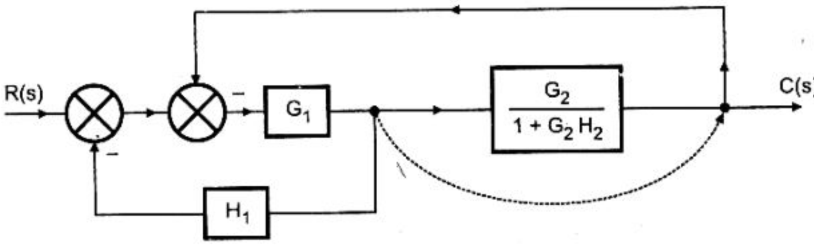
transfer function $\dfrac{C(s)}{R(s)}$ of the overall system.

Example 1 :

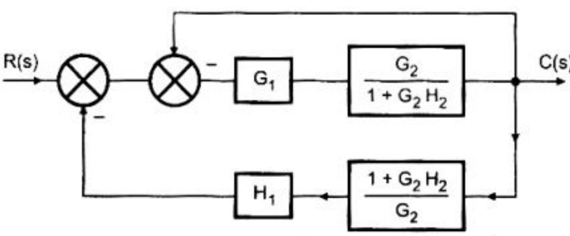


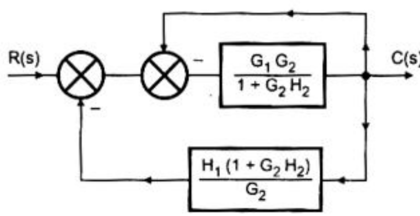
Solution :

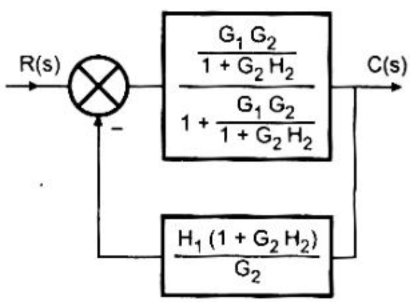
We can eliminate the minor loop of G_2 and H_2.

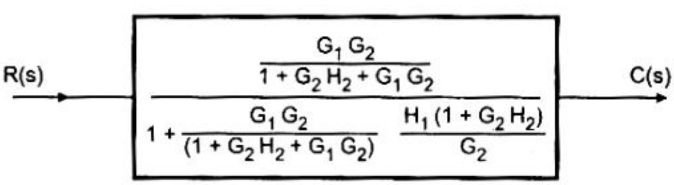


Always try to shift take off point towards left i.e. input.





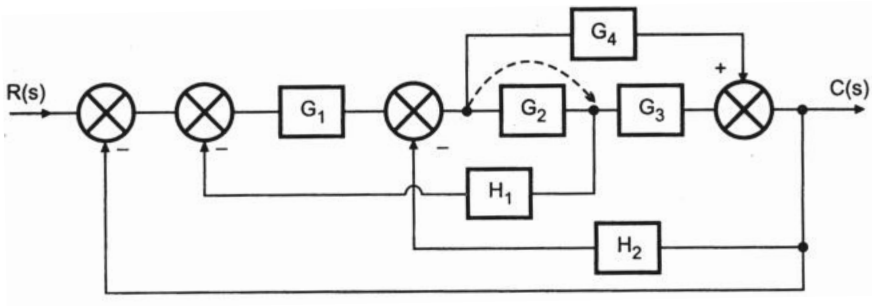




Simplifying

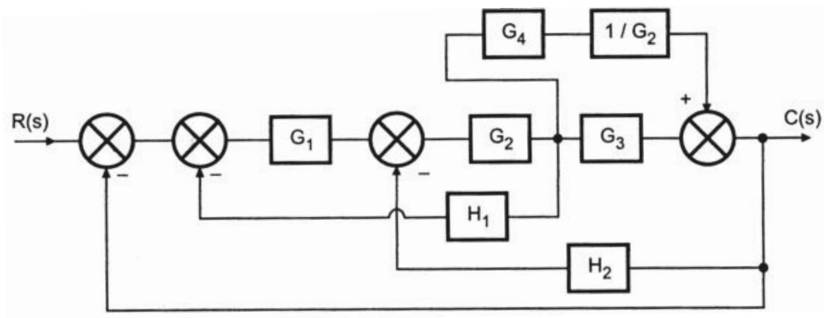
$$\frac{C(s)}{R(s)} = \frac{G_1G_2}{1 + G_1G_2 + G_2H_2 + G_1H_1 + G_1G_2H_1H_2}$$

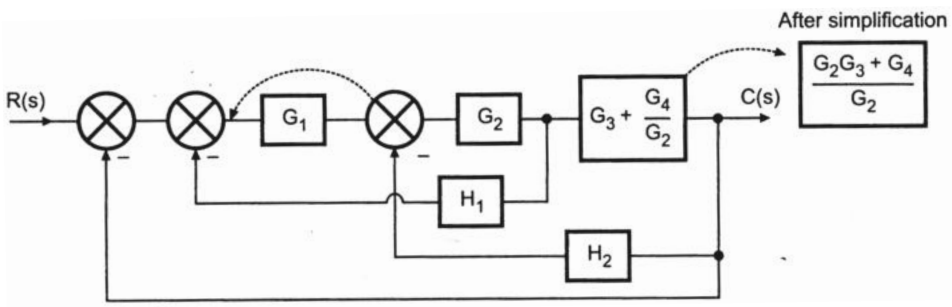
Example 2 :



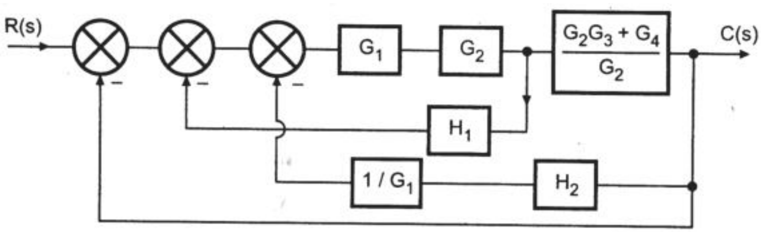
Solution :

Shifting take off point after the block having transfer function G_2 we get,

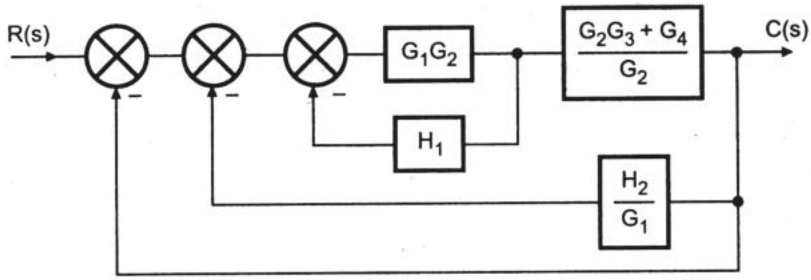


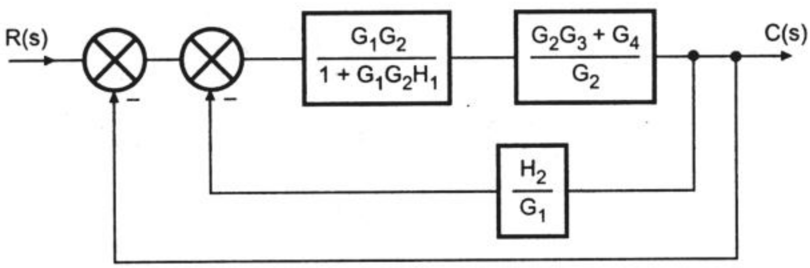


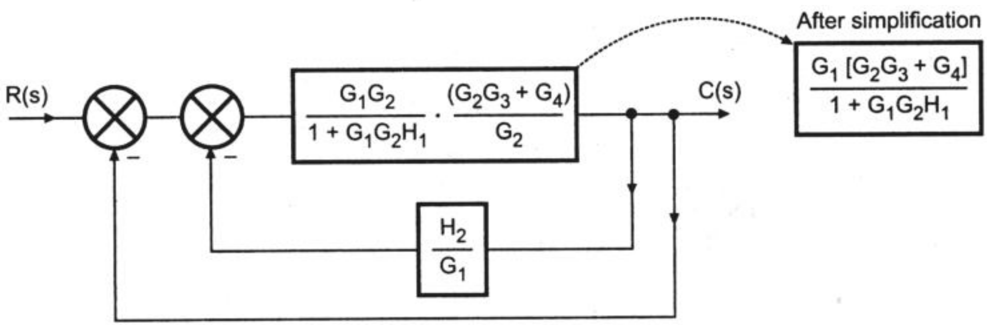
Shifting summing point before the block with transfer function 'G_1', we get,

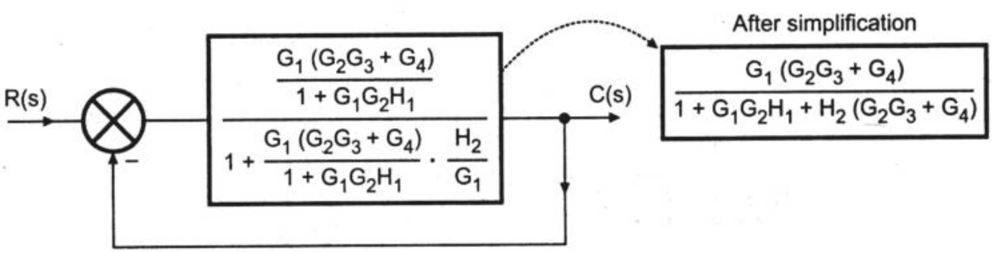


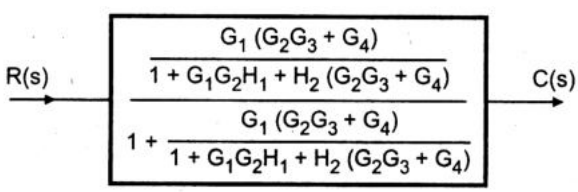
Using associative law for the summing points and interchanging their positions we get,





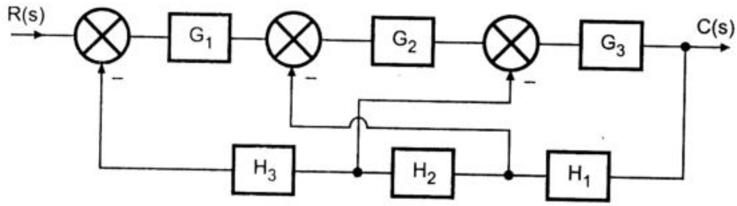






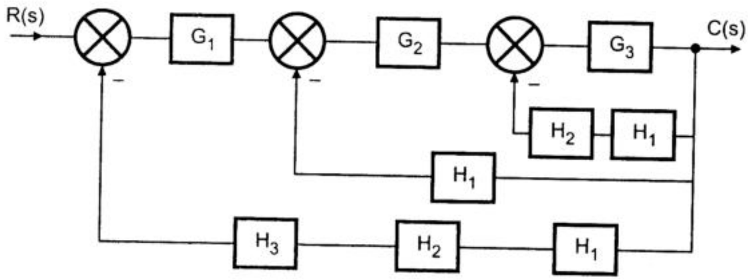
$$\therefore \qquad \frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + H_2G_2G_3 + H_2G_4 + G_1G_2G_3 + G_1G_4}$$

Example 3 :

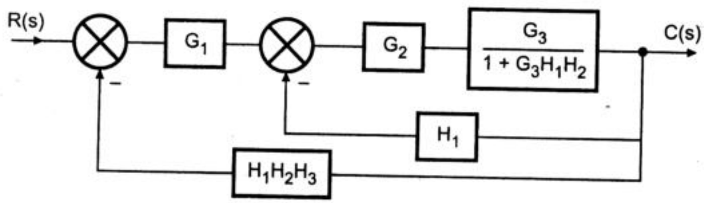


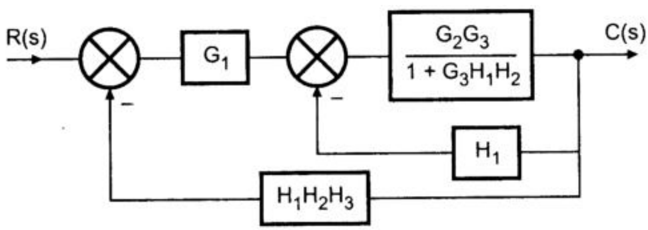
Solution :

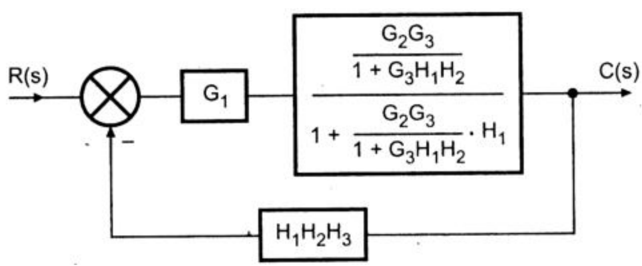
Separating out the feedbacks at different summing points, we can rearrange the above block diagram as below :

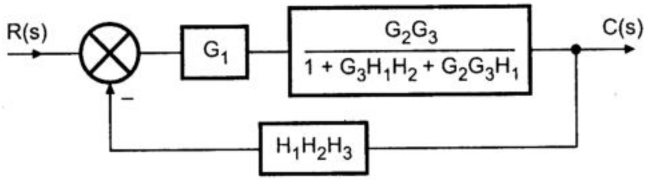


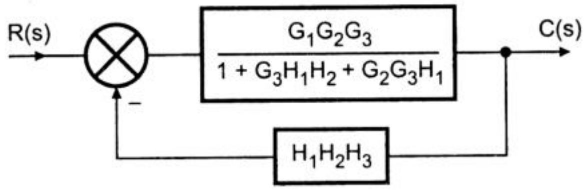
Solving minor feedback loop :

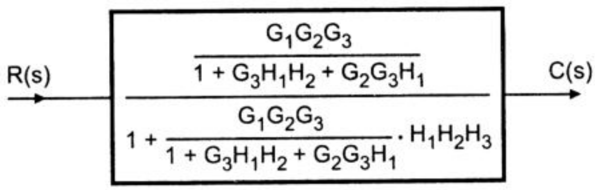










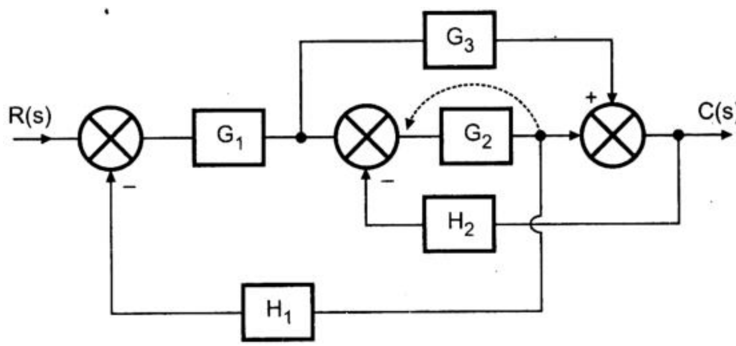


$$\frac{\dfrac{G_1G_2G_3}{1+G_3H_1H_2+G_2G_3H_1}}{1+\dfrac{G_1G_2G_3}{1+G_3H_1H_2+G_2G_3H_1}\cdot H_1H_2H_3}$$

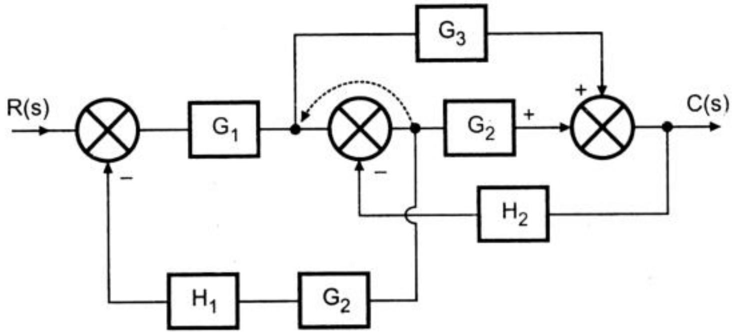
$$\therefore \qquad \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_3H_1H_2+G_2G_3H_1+G_1G_2G_3H_1H_2H_3}$$

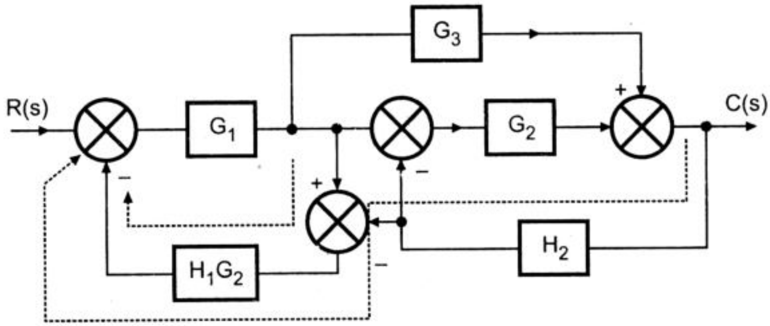
Example 4:

Use of Rule No. 10, critical rule illustration.

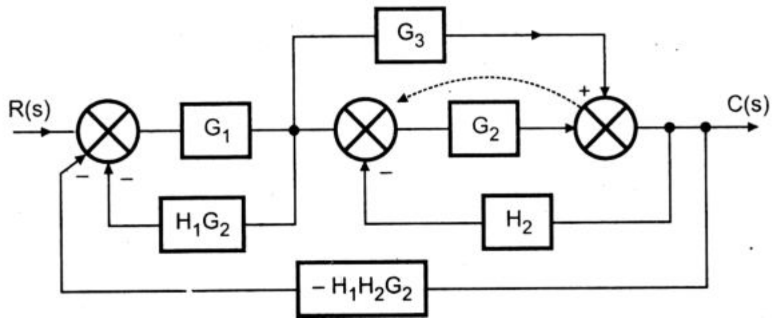


Solution :

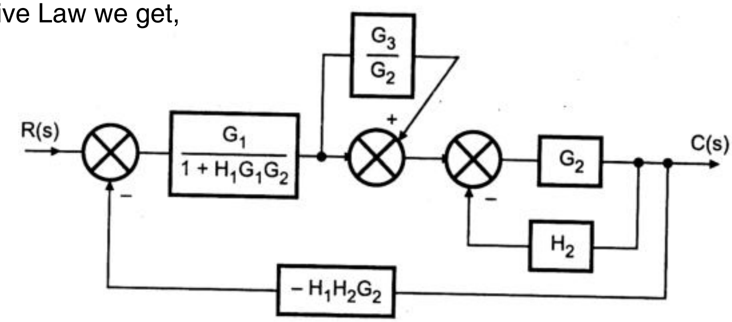


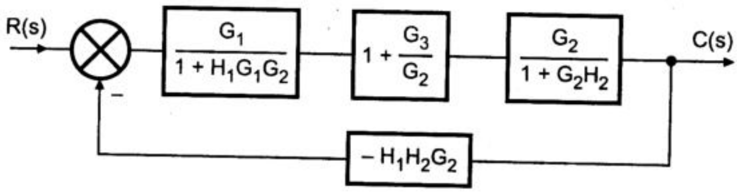


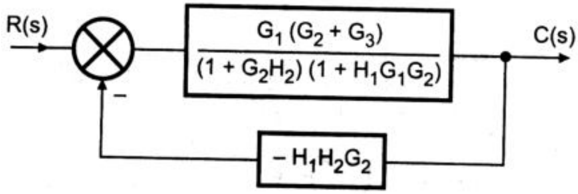
Separating the paths in the feedback path as shown



Shifting summing point as shown and then interchanging the two summing points using Associative Law we get,







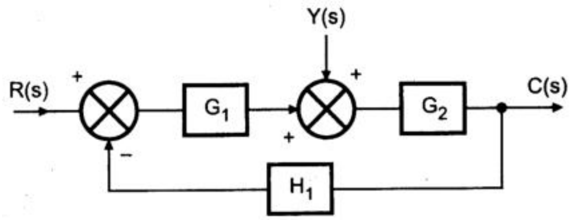
$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1(G_2+G_3)}{(1+G_2H_2)(1+H_1G_1G_2)}}{\dfrac{G_1(G_2+G_3)(-H_1H_2G_2)}{(1+G_2H_2)(1+H_1G_1G_2)}+1}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1(G_2+G_3)}{1+G_2H_2+H_1G_1G_2-G_1G_2G_3H_1H_2}$$

Multiple Input Multiple Output Systems

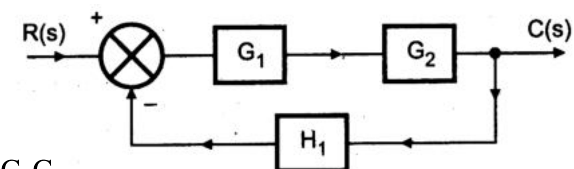
In multiple input system, each input is treated independently of others. Output of the system is obtained by superposition.

Let us consider following example



Output in terms of the inputs is obtained by considering one input at a time.

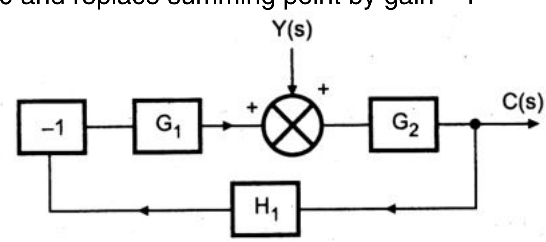
Consider Y(s) = 0

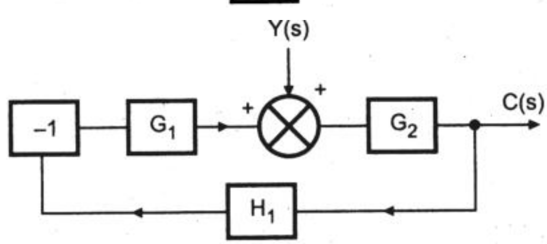


$$C(s) = \frac{G_1G_2}{1+G_1G_2H_1}R(s)$$

C(s) due to Y(s) only

Consider R(s) = 0 and replace summing point by gain '–1'





$$C(s) = \frac{G_2}{1-(-G_1G_2H_1)} = \frac{G_2}{1+G_1G_2H_1}Y(s)$$

$$C(s) = C_R(s) + C_Y(s)$$

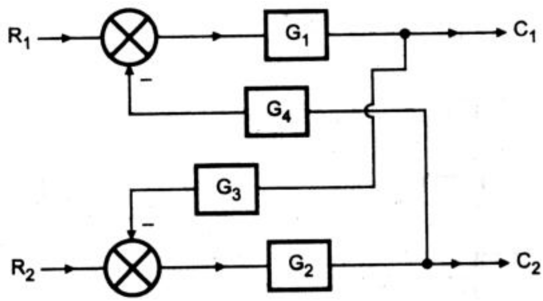
$$= \frac{G_2(s)}{1+G_1(s)\,G_2(s)\,H(s)}\left[G_1(s)R(s)+Y(s)\right]$$

Consider Y(s) is a disturbance input. Now if $|G_1(s)\ H(s)| >> 1$ and $|G_1(s)\ G_2(s)\ H(s)| >> 1$, then the closed loop transfer function $C_Y(s)/Y(s)$ becomes almost zero and the effect of disturbance is suppressed. This is one of the advantages of closed loop system.

On the other hand, as $|G_1(s)\ G_2(s)\ H(s)| >> 1$ then the closed loop transfer function C(s) /R(s) becomes independent of $G_1(s)$ and $G_2(s)$ and becomes inversely proportional to H(s). Thus by making H(s) = 1, closed loop system equalizes the input and output.

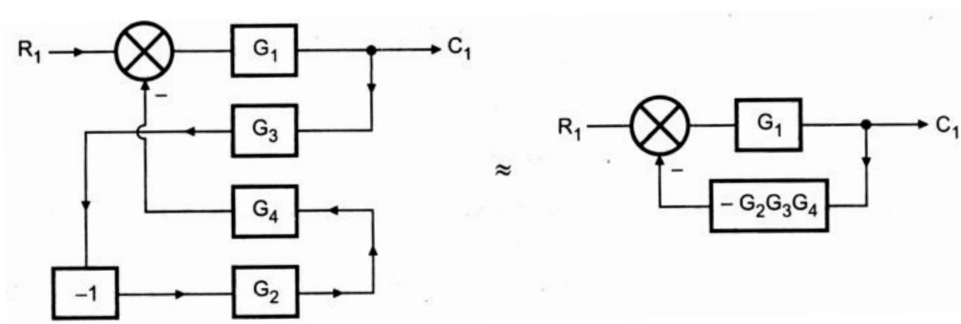
Example 5:

Obtain the expression for C_1 and C_2 for the given multiple input multiple output system.



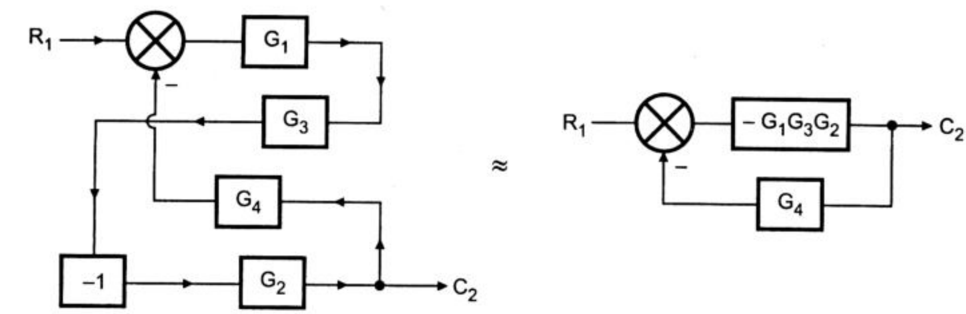
Solution:

In this case there are two inputs and two outputs. Consider one input at a time assuming other zero and one output at a time. Consider R_1 acting $R_2 = 0$ and C_2 not considered R_1, $R_2 = 0$ and C_2 is suppressed (not considered). C_2 suppressed does not mean that $C_2 = 0$. Only it is not the focus of interest while C_1 is considered. As $R_2 = 0$, summing point at R_2 can be removed but block of '−1 must be introduced in series with the signal which is shown negative at that summing point.



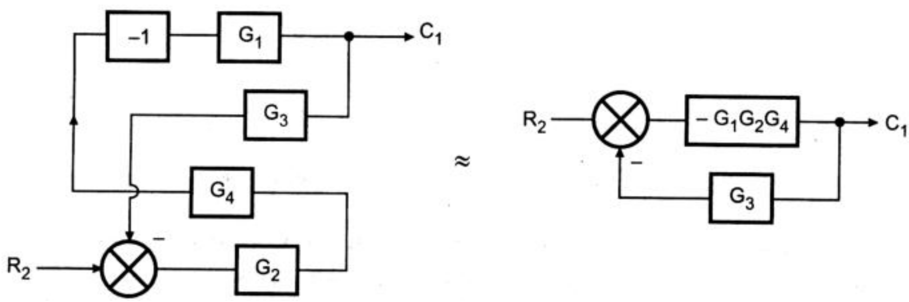
$$\frac{C_1}{R_1} = \frac{G_1}{1+[G_1][-G_2G_3G_4]} = \frac{G_1}{1-G_1G_2G_3G_4}$$

For $\frac{C_2}{R_1}$, assume C_1 suppressed.



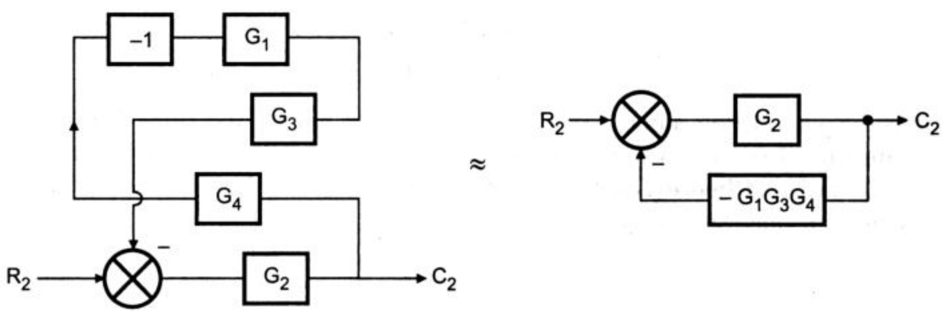
$$\therefore \qquad \frac{C_2}{R_1} = \frac{-G_1G_3G_2}{1+[-G_1G_3G_2][G_4]} = \frac{-G_1G_2G_3}{1-G_1G_2G_3G_4}$$

For $\frac{C_1}{R_2}$, $R_1 = 0$ and C_2 is suppressed.



$$\therefore \qquad \frac{C_1}{R_2} = \frac{-G_1G_2G_4}{1+[-G_1G_2G_4][G_3]} = \frac{-G_1G_2G_4}{1-G_1G_2G_3G_4}$$

For $\frac{C_2}{R_2}$, $R_1 = 0 \text{ and } C_1$ is suppressed.

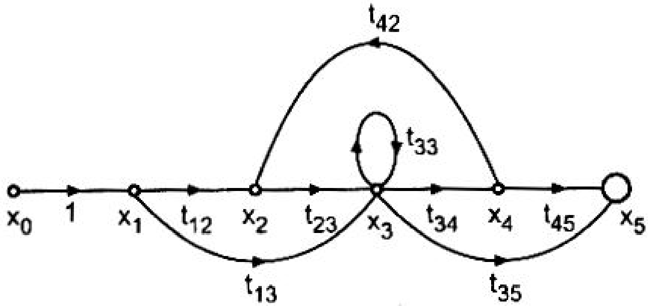


$$\frac{C_2}{R_2} = \frac{G_2}{1+[G_2][-G_1G_3G_4]} = \frac{G_2}{1-G_1G_2G_3G_4}$$

SIGNAL FLOW GRAPH (SFG) REPRESENTATION

Important Terms In SFG :

Consider a signal flow graph shown below :



i) Source Node :

The node having only outgoing branches is known as source or input node.
eg. X_0 is source node.

ii) Sink Node :

The node having only incoming branches is known as sink or output node.

iii) Chain Node :

A node having incoming and outgoing branches is known as chain node.
eg. $x_1, x_2, x_3 \text{ and } x_4$

iv) Forward Path :

A path from the input to output node is defined as forward path.

eg. $x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_5$ → 1st forward path

$x_0 \to x_1 \to x_3 \to x_4 \to x_5$ → 2nd forward path

$x_0 \to x_1 \to x_3 \to x_5$ → 3rd forward path

$x_0 \to x_1 \to x_2 \to x_3 \to x_5$ → 4th forward path

No node is to be traced twice

v) Feedback Loop :

A loop which originates and terminates at the same node is known as feedback path i.e. $x_2 \to x_3 \to x_4 \to x_2$ .No node is to be traced twice.

vi) Self Loop :

A feedback loop consisting of only one node is called self loop.
i.e. t_{33} at x_3 is self loop.

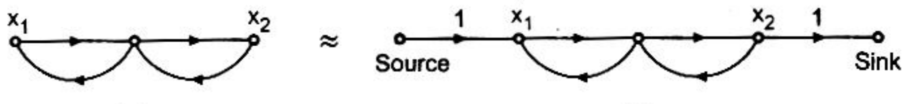
A self loop cannot appear while defining a forward path or feedback path as node containing it gets traced twice which is not allowed.

vii) Path Gain :

The product of branch gains while going through a forward path is known as path gain i.e. path gain for path $x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_5$ is, $1 \cdot t_{12} \cdot t_{23} \cdot t_{34} \cdot t_{45}$
This can also be called forward path gain.

viii) Dummy Node :

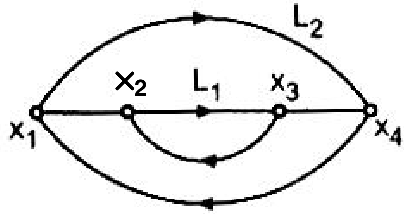
If there exists incoming and outgoing branches both at 1st and last node, representing input and output variables, then as per definition these cannot be called as source or sink nodes. In such a case separate input and output nodes can be created by adding branches with gain 1. Such nodes are called as Dummy nodes
eg.



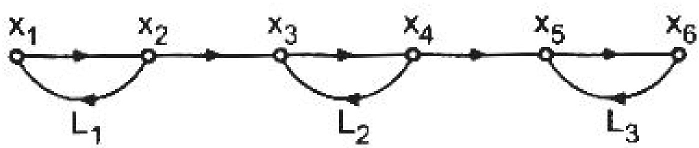
(a) (b)

ix) Non–touching Loops :

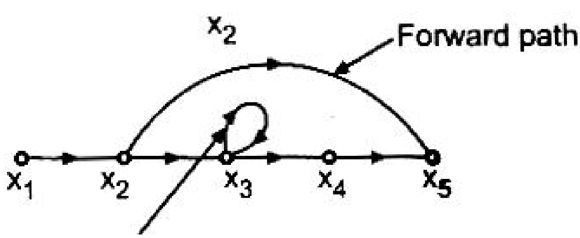
If there is no node common in between the two or more loops, such loops are said to be non–touching loops.



(a) Two non touching loops



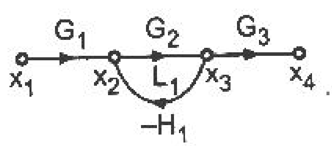
(b) Three non touching loops



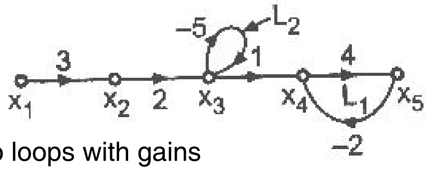
(c) Self loop non touching to forward path shown

x) Loop Gain :

The product of all the gains of the branches forming a loop is called as loop gain. For a self loop, gain indicated along it is its gain. Generally, such loop gains are denoted by 'L'.
eg. L_1, L_2 etc.



In the above figure, there is one loop with gain $L_1 = G_2 \times - H_1$



Here there are two loops with gains

$L_1 = 4 \times -2$

$L_1 = -8$

and other self loop with $\boxed{L_2 = -5}$

METHODS TO OBTAIN SIGNAL FLOW GRAPH :

- **From System Equations :**

  Steps :

  1) Represent each variable by a separate node
  2) Use the property that value of the variable represented by a node is an algebraic sum of all the signals entering at that node, to simulate the equations.
  3) Coefficients of the variables in the equations are to be represented as the branch gains, joining the nodes in signal flow.
  4) Show the input and output variables separately to complete signal flow graph.

     Example :

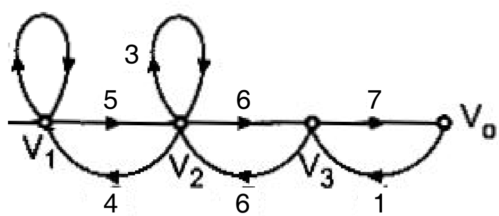
     Consider the following system equations :

     $V_1 = 3\,V_1 + 4\,V_2$

     $V_2 = 5\,V_1 + 6\,V_3 + 3\,V_2$

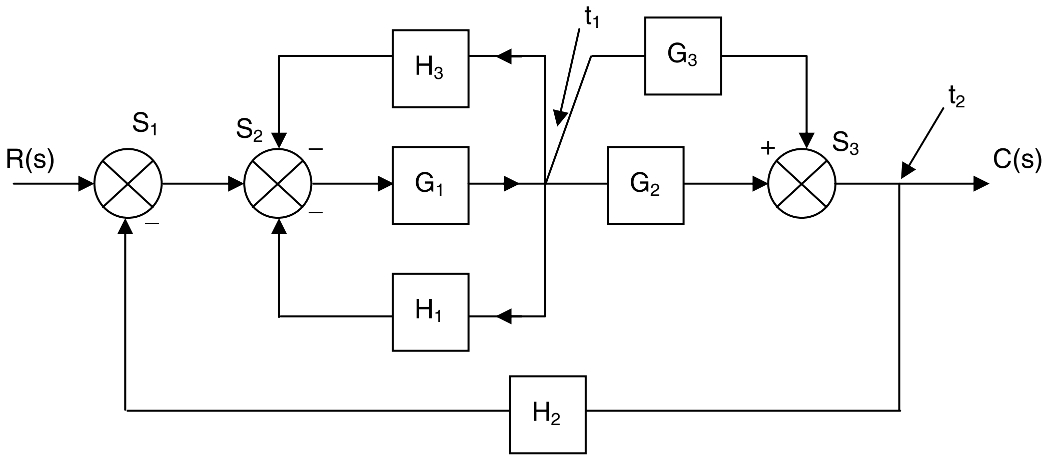
     $V_3 = 6\,V_2 + V_0$

     $V_0 = 7\,V_3$

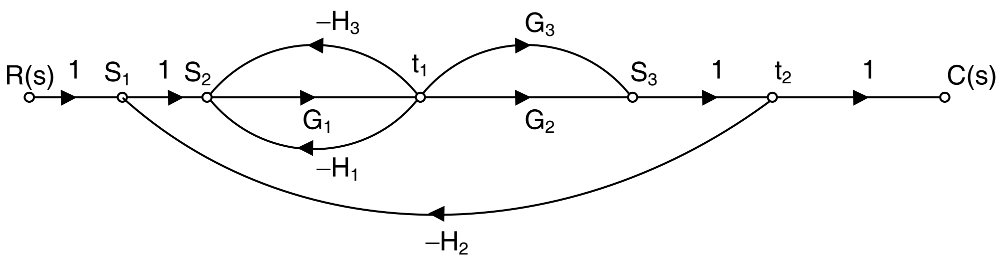


- **From given block diagram :**

  1) Name all the summing points and take off points in the block diagram.
  2) Represent each summing and take off point by a separate node in signal flow graph.
  3) Connect them by the branches instead of blocks, indicating block transfer functions as the gain of the corresponding branches.
  4) Show the input and output nodes separately if required to complete signal flow graph.



The complete SFG for the above block diagram is



- **Mason's Gain Formula :**

In signal flow graph approach, once SFG is obtained direct use of Mason's gain formula leads to the overall system transfer function $\frac{C(s)}{R(s)}$.

The formula can be stated as :

Overall transfer function (T.F) = $\dfrac{\sum_K T_K \Delta_K}{\Delta}$

where, k = Number of forward paths

T_k = Gain of k^{th} forward path

Δ = 1 – [Σ all individual feedback loop gains(including self loops)] +

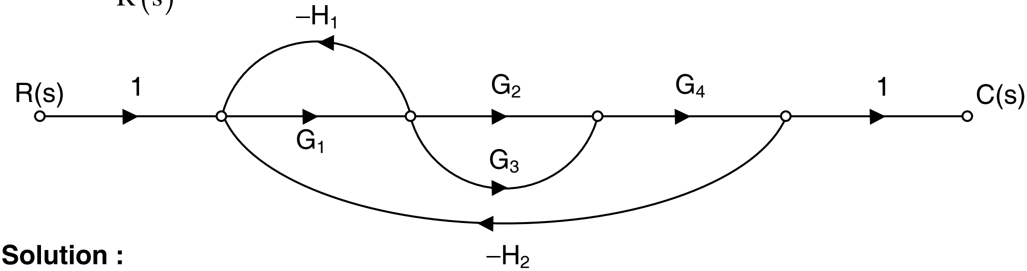
[Σ Gain product of all possible combinations of two non touching loops] –

[Σ Gain product of combination of three non touching loops] + …

Δ_k = Value of above Δ by eliminating all loop gains and associated product which are touching to the k^{th} forward path.

Example 1 :

Find $\frac{C(s)}{R(s)}$



Solution :

Number of forward paths = k = 2

By Mason's Gain formula,

$$\text{T.F.} = \sum_{k=1}^{2} \frac{T_k \Delta_k}{\Delta} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$T_1 = G_1\, G_2\, G_4$

$T_2 = G_1\, G_3\, G_4$

Individual feedback loops :

$L_1 = -\, G_1\, H_1$

$L_2 = -\, G_1\, G_2\, G_4\, H_2$

$L_3 = -\, G_1\, G_3\, G_4\, H_2$

$\Delta = 1 - [L_1 + L_2 + L_3]$ All loops are touching

$\Delta = 1 + G_1\, H_1 + G_1\, G_2\, G_4\, H_2 + G_1\, G_3\, G_4\, H_2$

Consider $T_1 = G_1\, G_2\, G_4$ All loops are touching $\Delta_1 = 1$

$T_2 = G_1\, G_3\, G_4$ All loops are touching $\Delta_2 = 1$

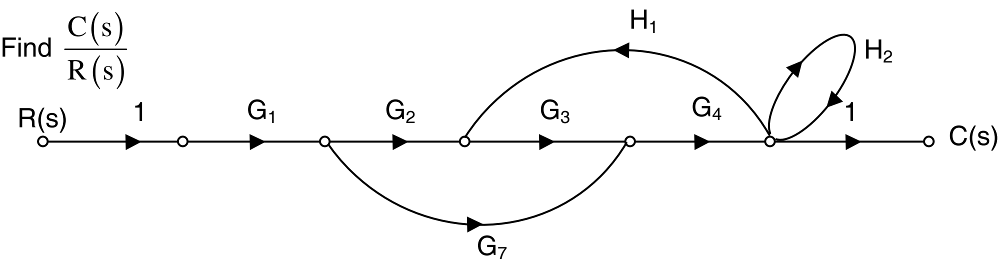
$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_4 \cdot 1 + G_1G_3G_4 \cdot 1}{\Delta}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 + G_1H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}}$$

Example 2 :

Find $\frac{C(s)}{R(s)}$



Solution :

Number of forward paths = k = 2

$$\text{T.F.} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$T_1 = G_1\, G_2\, G_3\, G_4$ and

$T_2 = G_1\, G_4\, G_7$

Individual loops

$L_1 = G_3\, G_4\, H_1$

$L_2 = H_2$

No combination of non–touching loops

$\Delta = 1 - [L_1 + L_2]$

$\Delta = 1 - G_3\, G_4\, H_1 - H_2$

For T_1,

$\Delta_1 = 1$

and for T_2,

$\Delta_2 = 1$

$$\frac{C(s)}{R(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4 + G_1G_4G_7}{\Delta}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_1G_4G_7}{1 - G_3G_4H_1 - H_2}$$

- **Application of Mason's Gain Formula to Electrical Network :**

The steps involved to solve electrical networks is as follows :

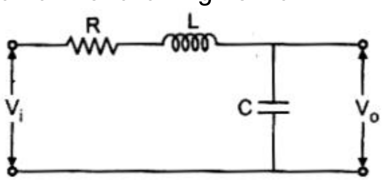
1. Find out Laplace transform of the given network and redraw the network in S–domain.
2. Write down the equations for the different branch currents and node voltages.
3. Simulate each equation by drawing corresponding signal flow graph.
4. Combine all signal flow graphs to get total signal flow graph for the given network.
5. Use Mason's Gain Formula to derive the transfer function of the given network.

- **Difference between Block Diagram and Signal Flow Graph :**

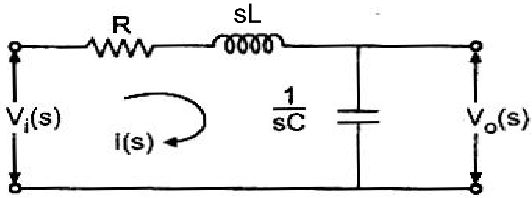
Sr. No.	Block Diagram	Signal Flow Graph
1.	Basic importance given is to the elements and their transfer functions.	Basic importance given is to the variables of the systems.
2.	Each element is represented by a block	Each variable is represented by a separate node
3.	Transfer function of the element is shown inside the corresponding block.	The transfer function is shown along the branches connecting the nodes.
4.	Summing points and takeoff points are separate.	Summing and takeoff points are absent. Any node can have any number of incoming and outgoing branches.
5.	Feedback path is present from output to input.	Instead of feedback path, various feedback loops are considered for the analysis.
6.	For a minor feed back loop present, the formula $\frac{G}{1 \pm GH}$ can be used.	Gains of various forward paths and feedback loops are just the product of associative branch gains. No such formula $\frac{G}{1 \pm GH}$ is necessary.
7.	Block diagram reduction rules can be used to obtain the resultant transfer function.	The Mason's gain formula is available which can be used directly to get resultant transfer function without reduction of signal flow graph.
8.	Method is slightly complicated and time consuming as block diagram is required to be drawn time to time after each step of reduction.	No need to draw the signal flow graph again and again. Once drawn, use of Mason's gain formula gives the resultant transfer function.
9.	Concept of self loop is not existing in block diagram approach.	Self loops can exist in signal flow graph approach.
10.	Applicable only to linear time invariant systems.	Applicable to linear time invariant systems.

Example 1:

Find the transfer function of the following network



Solution :



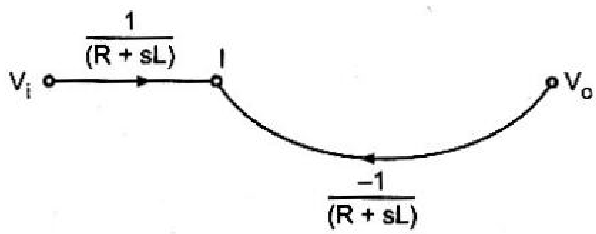
Assume different branch currents as shown

$$I(s) = \frac{(V_i - V_o)}{(R + sL)} \qquad \ldots(1)$$

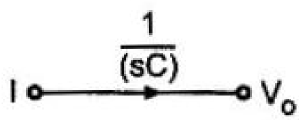
$$V_0(s) = I \times \frac{1}{sC} \qquad \ldots(2)$$

Now let us draw signal flow graph for the above 2 equations

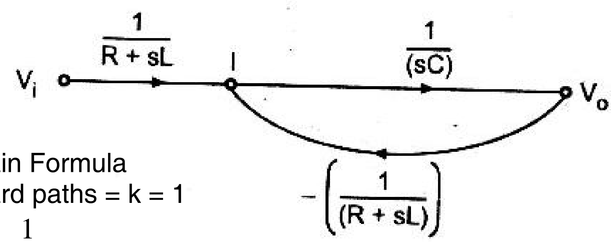
For equation (1) :



For equation (2) :



Combining the above two graphs total signal flow graph is



Use Mason's Gain Formula

Number of forward paths = k = 1

$$T_1 = \frac{1}{sC(R + sL)}$$

Individual loops

$$L_1 = -\frac{1}{sC(R + sL)}$$

$$\Delta = 1 - [L_1]$$

$$= 1 + \frac{1}{sC(R + sL)} \quad = \frac{1 + sRC + s^2LC}{sC(R + sL)}$$

For T_1,

L_1 is touching

$\therefore \quad \Delta_1 = 1$

$$\therefore \quad \frac{V_0(s)}{V_i(s)} = \frac{T_1\Delta_1}{\Delta}$$

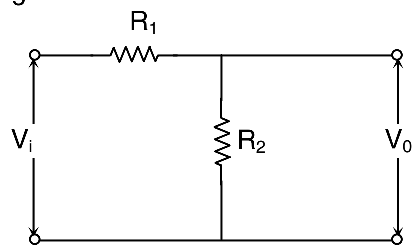
$$= \frac{\dfrac{1}{sC(R+sL)\cdot}}{\Delta}$$

$$\Delta = \frac{\dfrac{1}{sC(R+sL)}}{\dfrac{1+sRC+s^2RC}{sC(R+sL)}}$$

$$\therefore \quad \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2LC+sRC+1}$$

Example 2:

Find transfer function of the given network



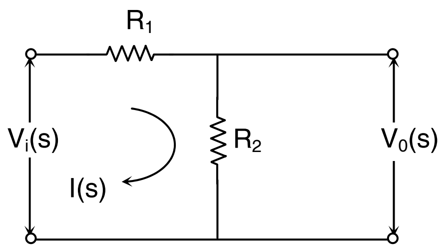
Solution :

Laplace transform of the given network is

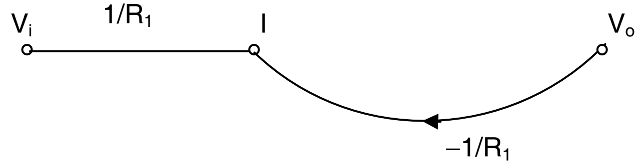
$$I(s) = \frac{V_i - V_o}{R_1} \qquad \ldots(1)$$

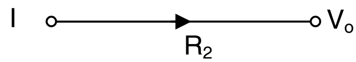
$$V_o(s) = I(s)R_2 \qquad \ldots(2)$$

For equations (1)

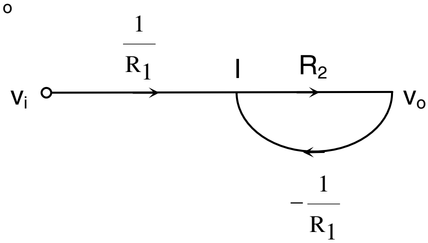


For equation (2)





$\therefore$ Combing two graphs, we get



Use Mason's gain formula,

Number of forward paths = k = 1

$$T_1 = \frac{R_2}{R_1}$$

$$L_1 = -\frac{R_2}{R_1}$$

$$\Delta = 1 - [L_1]$$

$$= 1 + \frac{R_2}{R_1}$$

$$= \frac{R_1 + R_2}{R_1}$$

As L_1 is touching to T_1, $\Delta_1 = 1$

$$\therefore \quad \frac{V_o(s)}{V_i(s)} = \frac{T_1\Delta_1}{\Delta} = \frac{\frac{R_2}{R_1}}{\frac{R_1 + R_2}{R_1}}$$

$$\therefore \quad \frac{V_o(s)}{V_i(s)} = \frac{R_2}{R_1 + R_2}$$

LIST OF FORMULAE

- For a closed loop systems having G(s) as the forward path gain and H(s) as the feedback factor, the transfer function is given as

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

+ sign $\rightarrow$ negative feedback
– sign $\rightarrow$ positive feedback

- In signal flow graph approach, once SFG is obtained direct use of Mason's gain formula leads to the overall system transfer function $\frac{C(s)}{R(s)}$.

  The formula can be stated as :

  Overall transfer function (T.F) = $\frac{\sum T_K \Delta_K}{\Delta}$

  where,

  k = Number of forward paths

  T_k = Gain of k^{th} forward path

  Δ = 1 – [Σ all individual feedback loop gains(including self loops)] +

  [Σ Gain × Gain product of all possible combinations of two non touching loops] –

  [Σ Gain × Gain × Gain product of combination of three non touching loops] + …

  Δ_k = Value of above Δ by eliminating all loop gains and associated product which are touching to the k^{th} forward path.

LMR(LAST MINUTE REVISION)

- The transfer function is defined as the ratio of Laplace transform of output to Laplace transform of input under assumption that all initial conditions are zero

- The stability of a time–invariant line system can be determined from the characteristic equation. Consequently, for continuous systems, if all the roots of the denominator have negative real parts, the system is stable.

- The system differential equation can be obtained from the transfer function by eplacing the s variable with d/dt.

- Transfer function is valid only for linear time invariant system.
- The value of s for which the system magnitude | G(s) | becomes infinity are called poles of G(s). When pole values are not repeated, such poles are called as simple poles. If repeated such poles are called multiple poles of order equal to the number of times they are repeated.

- The value of s for which the system magnitude | G(s) | becomes zero are called zeros of transfer function G(s). When they are not repeated, they are called simple zero, otherwise they are called multiple zeros.

- In Block diagram reduction, gain of the blocks in series gets multiplied whereas that of in parallel gets added or subtracted depending upon the sign of the summer.

- Signal flow graph specifications :

  i) The node having only outgoing branches is known as source or input node.

  ii) The node having only incoming branches is known as sink or output node.

  iii) A node having incoming and outgoing branches is known as chain node.

  iv) A path from an input to an output node is defined as forward path.

  v) A loop which originates and terminates on the same node is known as feedback path.

  vi) A feedback loop consisting of only one node is called self loop.

  vii) A self loop cannot appear while defining a forward path or feedback path as node containing it gents traced twice which is not allowed.

  viii)The product of branch gains while going through a forward path is known as path gain. This can also be called forward path gain.

  ix) If there exists incoming and outgoing branches both at 1^{st} and last node, representing input and output variables, then as per definition these cannot be called as source or sink nodes. In such a case separate input and output nodes can be created by adding branches with gain 1. Such nodes are called as Dummy nodes

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Topic 3 : Time Response

The Response of the system as a function of time, to the applied excitation is called Time Response. The time response of a system can be fully studied by studying the following two responses

Time Response

- Transient Response (function of time called as time domain response)
- Steady State Response

Mathematically,

The total time response C(t) is given by

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

where $C_{tr}(t)$ = transient response

$C_{ss}(t)$ = steady state response

DIFFERENTIAL EQUATION

Consider the class of differential equation

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^i}$$

where the coefficient a_i and b_i are constant, u = u(t) (the input) is a known time function, and y = y(t) (the output) is the unknown solution of the equation. Generally $m \le n$, and n is called the **order** of the differential equation.

The Free Response

The Free response of a differential equation is the solution of the differential equation when the input u(t) is identically zero.

If the input u(t) is identically zero, then the differential equation has the form :

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = 0 \qquad \text{....(1)}$$

The solution y(t) of such an equation depends only on the initial conditions.

The Forced Response

The forced response $y_b(t)$ of a differential equation is the solution of the differential equation when all the initial conditions are identically zero.

$$y(0), \left.\frac{dy}{dt}\right|_{t=0}, \ldots\ldots, \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=0}$$

The Total Response

The total response of a linear constant –coefficient differential equation is the sum of the free response and the forced response.

The Steady State and Transient Response

The steady state response and transient response are another pair of quantities whose sum is equal to the total response. These terms are often used for specifying control system performance. They are defined as follows.

> The steady state response is that part of the total response which does not approach zero as time approaches infinity.

> The transient response is that part of the total response which approaches zero as time approaches infinity.

Mathematically, for stable systems

$$\boxed{\lim_{t\to\infty} C_{tr}(t) = 0}$$

> For stable system Transient response vanishes after some time to get final value closer to the desired value.

Steady State Response :

The part of the time response which remains after complete transient response vanishes from the system output. The steady state response is generally the final value achieved by the system output.

The difference between the desired output and the actual output of the system is called as Steady state error (e_{ss}) and the time taken to reach steady state is called Settling time.

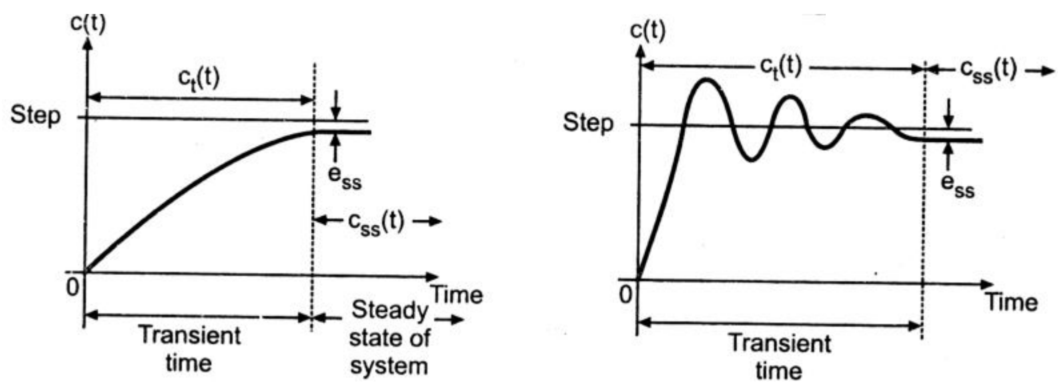


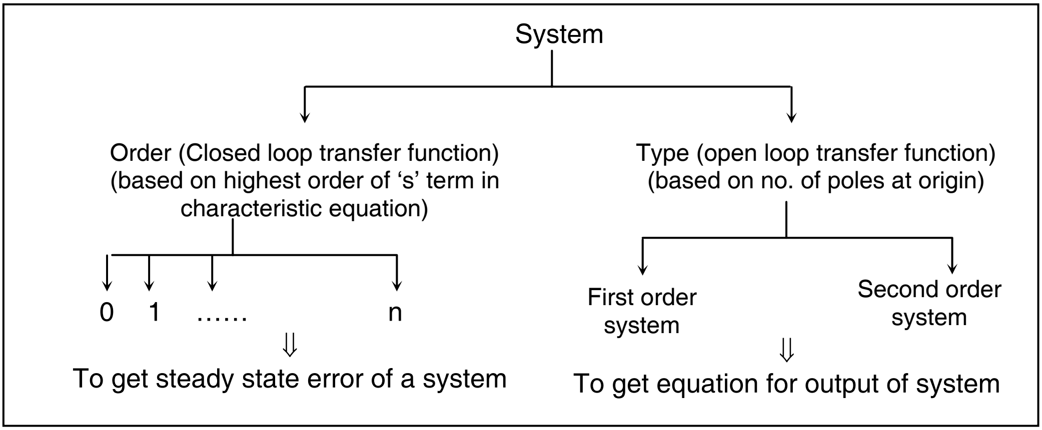
Fig. (a) $C_t(t)$ is exponential (b) $C_t(t)$ is oscillatory

To study response of a system completely, we should have Mathematical model.

Mathematical model of a system :

This is usually the differential equation description of the system. The differential equation is converted to Laplace domain to simplify analysis.

Mathematical representation of system inputs :



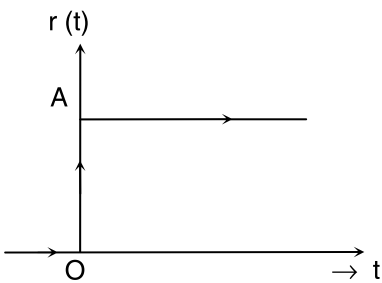
STANDARD TEST INPUTS

In practice, many signals are available which are the functions of time and can be used as reference inputs for the various control systems. These signals are step, ramp, saw tooth type, square wave, triangular etc. But while analysing the system it is highly impossible to consider each one of it as input and study the response. Hence from the analysis point of view, those signals which are most commonly used as a reference inputs are defined as Standard Test Inputs. The evaluation of the system can be done on the basis of the response given by the system to the standard test inputs. Once system behaves satisfactorily to the test input, its time response to actual input is assumed to be upto the mark.

These standard test signals are :

1. Step Input [Position function]

The step is a signal whose value changes from one level (level) to another level (A)in zero time. Mathematically it can be represented as



$r(t) = A$ For $t \geq 0$
$= 0$ For $t \leq 0$

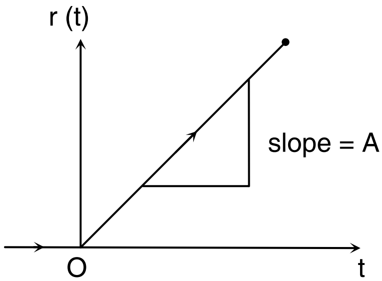
If $A = 1$ then it is called as unit step function and denoted by $u(t)$

Laplace transform of such input is

$$\boxed{r(S) = A/S}$$

2. Ramp Input [Velocity function]

The ramp is a signal which start at a value of zero and increases linearly with time.



Mathematically it is defined as

$r(t) = A\,t$ for $t \geq 0$

$r(t) = 0$ for $t < 0$

If $A = 1$

it is called unit ramp input.

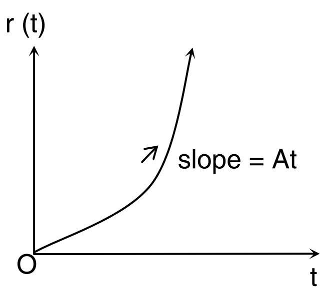
Its Laplace transform is $r(s) = A/s^2$

So ramp signal is Integral of step signal.

> Ramp signal is integral of step signal.

3. Parabolic Input [Acceleration function]

This is the input which is one degree faster than a ramp type of input, as shown in following figure



Mathematically it is denoted as

$$r(t) = \frac{At^2}{2} \quad \text{for } t \geq 0$$

$$= 0 \qquad \text{for } t < 0$$

If $A = 1$, $r(t) = \dfrac{t^2}{2}$ is called unit parabolic input.

Its Laplace transformation is $r(S) = \dfrac{A}{S^3}$.

> Parabolic input is integral of ramp input.

4. Impulse Function

An impulse is a unit step of extremely large magnitude and infinitesimal duration. If we go on reducing the width of a pulse keeping the area constant, the pulse height will increase as duration becomes shorter so that as $t \to 0$, magnitude $\to \infty$.

i.e. the function is zero everywhere except at $t = 0$.

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

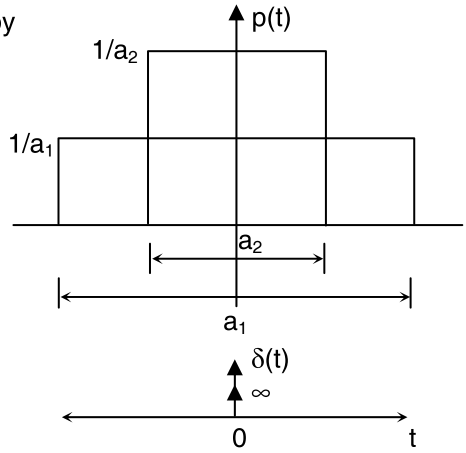
In Laplace domain,

$$L\{\delta(t)\} = 1$$

A unit impulse function $\delta(t)$ may be defined by

$$\delta(t) = \lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} \left[\frac{u(t) - u(t - \Delta t)}{\Delta t} \right]$$

where $u(t)$ is the unit step function.



The pair $\left\{ \begin{matrix} \Delta t \to 0 \\ \Delta t > 0 \end{matrix} \right\}$ may be abbreviated by $\Delta t \to 0^+$, meaning that Δt approaches zero from the right. The quotient in brackets represents a rectangle of height $1/\Delta t$ and width Δt as shown in fig. The area under the curve is equal to 1 for all values of Δt.

The unit impulse function has the following very important property :

Screening Property

The integral of the product of a unit impulse function $\delta(t - t_0)$ and a function f(t), continuous at $t = t_0$ over an interval which includes t_0, is equal to the function f(t) evaluated at t_0 that is,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

> The unit impulse response of a system is the output y(t) of the system when the input $u(t) = \delta(t)$ and all initial conditions are zero.

ANALYSIS OF FIRST ORDER SYSTEMS

A system which is described by a first order differential equation is a first order system. First order differential equation is of the form

$$f(t) = Ax + B\frac{dx}{dt}$$

Steady State or Forced Response

At steady state, all transients have died out, so output of the system resembles the input in steady state.

$\therefore \quad x = \text{constant}$

$$\frac{dx}{dt} = 0$$

$\therefore \quad f(t) = Ax$

$$x = \frac{f(t)}{A}$$

$$x_{ss} = \frac{f(t)}{A} = \frac{R}{A} \qquad \text{where} \quad R = \text{amplitude of step input}$$

Transient or Natural Response

The behaviour of the output from the initial value to the steady state value is called the transient response of the system.

$$Ax + B\frac{dx}{dt} = 0 \qquad \text{....(A)}$$

$$x = Be^{st}$$

$$\frac{dx}{dt} = sBe^{st} = sX$$

$$\frac{d^2x}{dt^2} = s^2x$$

∴ The total response is the sum of forced and transient responses

$$x = x_{ss} + x_{tr}$$

$$= \frac{R}{A} + Ce^{-At/B}$$

Note : $\frac{A}{B}$ determines rate of decay , $\frac{B}{A}$ is the time constant

$$\tau = \frac{\text{coefficient of time varying term}}{\text{coefficient of steady term}} = \text{Time constant}$$

It is the time in which transient reduces to e^{-1} = 0.368 (37 %) of its original value.

The smaller the time constant the faster the response.

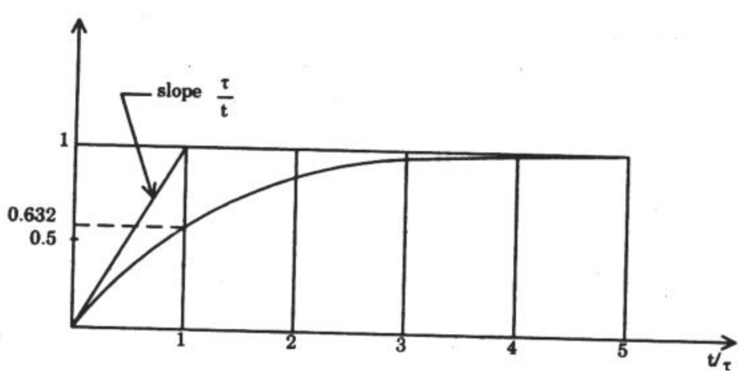
For a step input where initial condition is zero at t = 0, the total

$$x = x_{ss}\left(1 - e^{-t/\tau}\right)$$

The graph of step response of a first order system is shown. The x axis is t/τ in time constants. The y axis is the amplitude which is maximum of 1 for unit step.

For one time constant $\left(\frac{t}{\tau} = 1\right)$,

the response rises to 0.632 of final value.

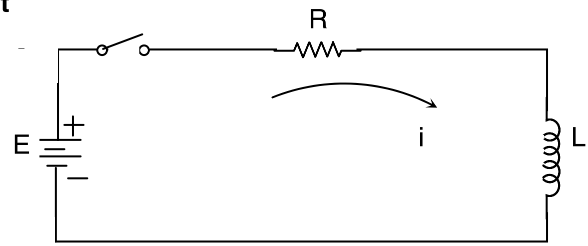


In 2 time constants – 0.865 or 86.5 %
In 3 time constants – 0.961 or 96.1 %
In 4 time constants – 0.981 or 98.1 %
In 5 time constants – 0.993 or 99.3 %

Thus, we see that, after 4 time constant the output remains within 2% of its final value.

Applications of First Order System

R–L Circuit

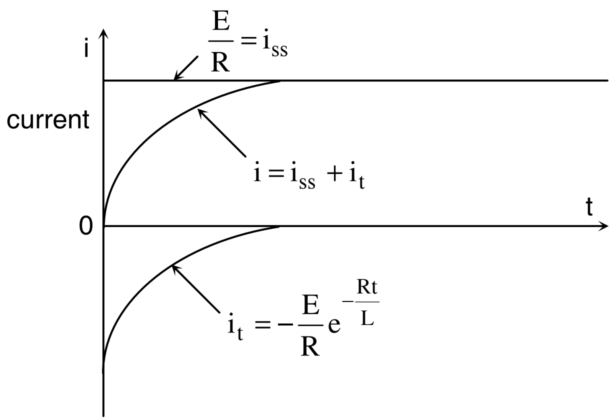


$E = Ri + L\frac{di}{dt}$. This is a first order differential equation similar to (1) and we can write the solution from equation to step input.

$$x = \frac{f(t)}{A}\left(1 - e^{-\frac{A}{B}t}\right)$$

$$i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

Time constant τ = B/A = L/R. $\frac{E}{R}$ is the steady state value and $-\frac{E}{R}e^{-Rt/L}$ is the transient current.



It may be seen from the diagram that at t = 1, the steady state value is equal to transient value but negative.

ANALYSIS OF SECOND ORDER SYSTEMS

Every practical system takes finite time to reach to its steady state and during this period it oscillates or increases exponentially. The behavior of system gets decided by type of closed loop poles and location of closed loop poles in s–plane. The closed loop poles are dependent on selection of the parameters of the system. Every system had tendency to oppose the oscillatory behavior of the system which is called as damping. This damping is measured by a factor or a ratio called as damping ratio of the system. This factor explains us, how much dominant the opposition is to the oscillations in the output. In some system it will be low in which case system will oscillate but slowly i.e., with damped frequency. Now as this measures the opposition by the system to the oscillatory behaviour, if it is made zero, ($\xi = 0$) system will oscillate with maximum frequency. As there is no opposition from system, system naturally and freely oscillates under such condition. Hence this frequency of oscillation under $\xi = 0$ condition is called as Natural frequency of oscillations of the system and denoted by the symbol ω_n rad/sec.

In the study of control system, linear constant–coefficient second–order differential equations of the form :

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2\, y = \omega_n^2\, u \qquad \text{....(1)}$$

are important because higher–order systems can often be approximated by second–order systems. The constant ζ is called the **damping ratio**, and the constant ω_n is called the **undamped natural frequency** of the system. The forced response of this equation for inputs u belonging to the class of singularity functions is of particular interest. That is, the forced response to a unit impulse, unit step, or unit ramp is the same as the unit impulse response, unit step response, or unit ramp response of a system represented by this equation.

Assuming that $0 \le \zeta \le 1$, the characteristic equation for equation (1) is

$$D^2 + 2\zeta\omega_n D + \omega_n^2 = \left(D + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)\left(D + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right) = 0$$

Hence the roots are

$$D_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -\alpha + j\omega_d \qquad D_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} = -\alpha - j\omega_d$$

where $\alpha \equiv \zeta\omega_n$ is called the **damping coefficient**,

and $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$ is called the **damped natural frequency**.

α is the inverse of the **time constant** τ of the system, that is, $\tau = 1/\alpha$.

The weighting function of equation (1) is $w(t) = (1/\omega_d)\, e^{-\alpha t} \sin \omega_d t$.

The unit step response is given by

$$y_1(t) = \int_0^t w(1-\tau)\,\omega_n^2\, d\tau = 1 - \frac{\omega_n e^{-\alpha t}}{\omega_d}\sin\left(\omega_d t + \phi\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

where $\phi \equiv \tan^{-1}(\omega_d / \alpha)\,. = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

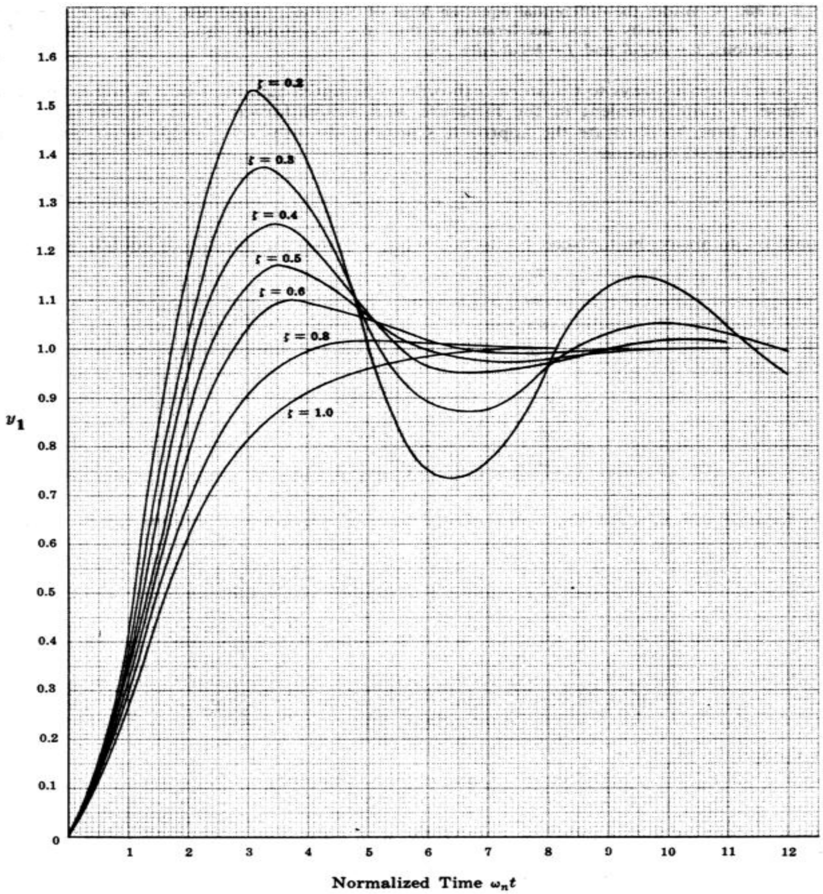
Figure below is a parametric representation of the unit step response. Note that the abscissa of this family of curves is normalized time $\omega_n t$, and the parameter defining each curve is the damping ratio ζ.

The Laplace transform of y(t), when the initial conditions are zero, is

$$Y(s) = \left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] U(s)$$

where $U(s) = L\,[u(t)]$. The poles of the function $Y(s)/U(s) = \omega_n^2 / \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$ are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



Note :

1. If $\zeta > 1$, both poles are negative and real.
2. If $\zeta = 1$, the poles are equal, negative, and real ($s = -\omega_n$)
3. If $0 < \zeta < 1$, the poles are complex conjugates with negative real parts ($s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$).
4. If $\zeta = 0$, the poles are imaginary and complex conjugate ($s = \pm j\omega_n$)
5. If $\zeta < 0$, the poles are in the right half of the s–plane (RHP).

Of particular interest representing an **underdamped second–order system**. The poles are complex conjugates with negative real parts and are located at

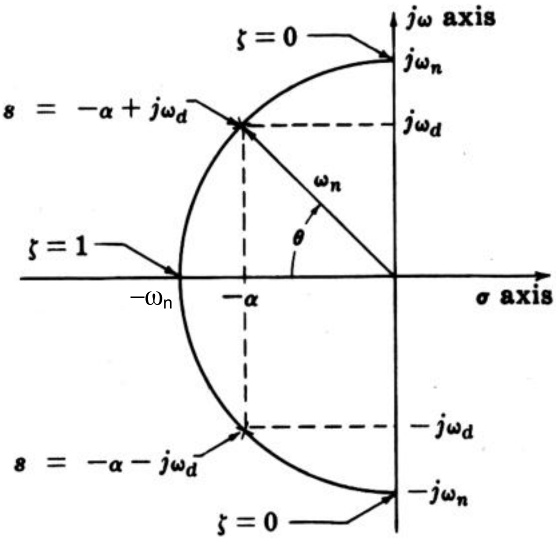
$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

or at $s = -\alpha \pm j\omega_d$

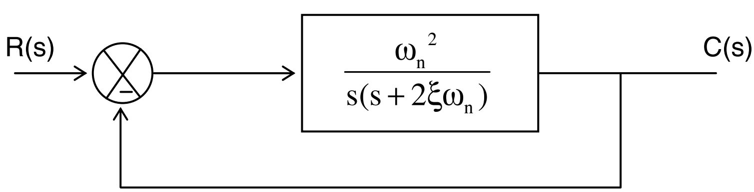
For fixed ω_n. Figure below shows the locus of these poles as a function of ζ, $0 < \zeta < 1$.

The locus is a semicircle of radius ω_n. The angle θ is related to the damping ratio by $\theta = \cos^{-1}\zeta$.

A similar description for second–order systems described by difference equations does not exist in such a simple and useful form.



Let us consider second order system :



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where ω_n = natural frequency, rad/sec, ξ = damping ratio.

Step Response Analysis of Second order system

Consider input applied to the standard second order system is unit step.

$$R(s) = \frac{1}{s}$$

while
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \left[\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] \times \frac{1}{s}$$

Finding the roots of the equation,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

i.e.,
$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

i.e.,
$$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

So we can write above C(s) equation as

$$C(s) = \frac{\omega_n^2}{s[s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}](s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}]}$$

Now, nature of these roots is depend on damping ratio ξ. Consider following cases :

Case I : $1 < \xi < \infty$ (overdamped)

The roots are

$$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

i.e., roots are real, unequal and negative.

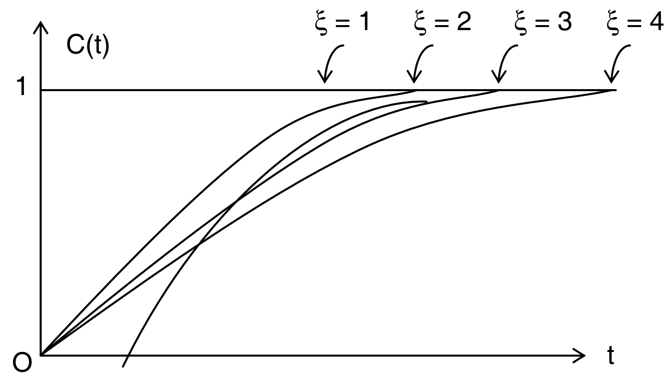
$$C(s) = \frac{\omega_n^2}{s(s + K_1)(s + K_2)} = \frac{A}{s} + \frac{B}{s + K_1} + \frac{C}{s + K_2}$$

Taking Laplace inverse

$$C(t) = A + B\,e^{-k_1 t} + C e^{-k_2 t}$$

$$C(t) = C_{SS}(t) + C_t(t)$$

which is purely exponential, this means damping is so high, that there are no oscillations in the output and is purely exponential. Hence such system are called as *overdamped.* Hence nature of response will be shown in fig.



As ξ increases, output will take more time to reach its steady state and hence become *sluggish* and *slow.*

Case II : $\xi = 1$ (Critically damped)

When $\xi = 1$, the roots are $S_{1,2} = -\omega_n, -\omega_n$.
i.e., real, equal and negative.

$$C(S) = \frac{\omega_n^2}{s(s+\omega_n)\ (s+\omega_n)} = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

Take partial fraction :

$$C(S) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{(s+\omega_n)}$$

Taking Laplace inverse, C(t) will take the following form.

$$C(t) = A + Bt\ e^{-\omega_n t} + Ce^{-\omega_n t}$$

$$C(t) = C_{SS}(t) + C_t(t)$$

This is purely exponential, but in comparison with *overdamped* case, settling required for this case is less. Because of repetitive occurrence of roots, the system is called as *Critically Damped* This is critical value of damping ratio because if it is decreased further roots will becomes complex conjugates and this is least value of damping ratio for which roots are real and negative. So $\xi = 1$ system critically damped and corresponding response is exponential.

Case III : $0 < \xi < 1$ (underdamped)

When range of ξ is $0 < \xi < 1$, the roots are,

$$s_{1,2} = -\xi\,\omega_n \pm j\,\omega_n\sqrt{1-\xi^2}$$

i.e., complex conjugates with negative real part.

$$C(s) = \frac{\omega_n^2}{s\left[s+\xi\omega_n - j\omega_n\sqrt{1-\xi^2}\right]\left[s+\xi\omega_n + j\omega_n\sqrt{1-\xi^2}\right]}$$

$$= \frac{\omega_n^2}{s[s^2+2\xi\omega_n s+\omega_n^2]} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

Take inverse Laplace

$$C(t) = A + K\ e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\quad t+s\right)$$

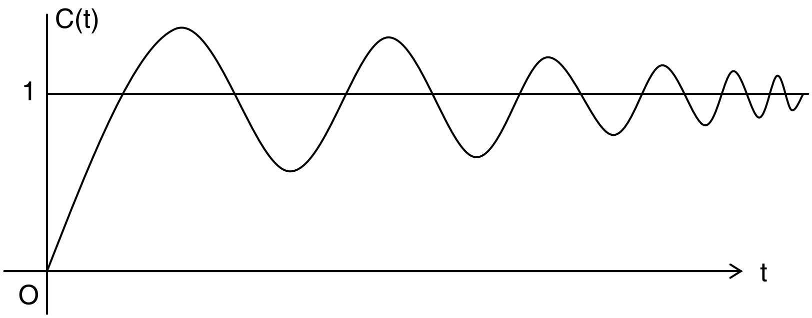
$$C(t) = C_{ss}(t) + C_t(t)$$

This response is oscillatory with oscillating frequency $\omega_n \sqrt{1-\xi^2}$ but decreasing amplitude as it is associated with exponential term with negative index $e^{-\xi\omega_n t}$. Such oscillations are called as Damped oscillations and frequency of such oscillations is called as *Damped frequency* of oscillation ω_d which is nothing but $\omega_n \sqrt{1-\xi^2}$.

i.e., $\boxed{\omega_d = \omega_n \sqrt{1-\xi^2}}$ rad/sec.

In such response, real part of complex roots controls the amplitude while imaginary part control the frequency of damped oscillations.

The response of such system is shown in following figure.



Case IV : ξ = 0 [undamped]

When $\xi = 0$, then roots are $S_{1,2} = \pm j\ \omega_n$.
i.e., complex conjugates with zero real parts i.e., purely imaginary.

$$C(S) = \frac{\omega_n^2}{s(s+j\omega_n)(s-j\omega_n)} = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

Take partial traction,

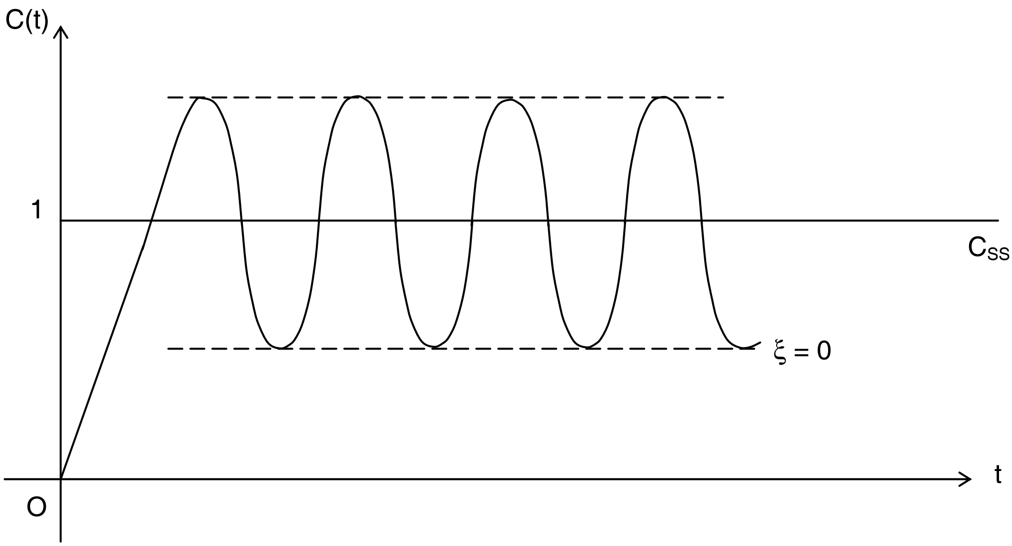
$$C(S) = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2}$$

Take inverse Laplace transform :

$$C(t) = A + K'' \sin(\omega_n t + \theta)$$

$$C(t) = C_{SS}(t) + C_t(t) \qquad \text{where } K'' = \text{constant.}$$

The response is purely oscillatory, oscillating with constant frequency and amplitude. The frequency of such oscillations is the maximum frequency with which output can oscillate. At this frequency is under the condition $\xi = 0$ i.e., no opposition condition system oscillates freely and naturally. Hence this frequency is called as natural frequency of oscillations denoted by ω_n rad/sec. The systems are classified as undamped systems. The response of such system is shown in following figure.



- **Summarizing all cases as is shown in following table :**

Sr. No.	Range of ξ	Types of closed loop poles	Nature of response	System classification
1	$1 < \xi < \infty$	Real, unequal & negative	Purely exponential	Overdamped
2	$\xi = 1$	Real, equal & negative	Critically pure exponential	Critically damped
3	$0 < \xi < 1$	Complex conjugate with negative real part.	Damped oscillations	Underdamped
4	$\xi = 0$	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped

- **Derivation of Unit Step Response of a second order system for underdamped Case ($0 < \xi < 1$)**

> For the critical and overdamped systems, calculation of partial fraction is not a difficult exercise and hence this derivation is strictly valid for underdamped system where calculation of partial fraction is slightly complicated. Hence this result can used as a standard result for underdamped systems.

For standard second order system T.F. is :

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For this $\xi < 1$ because we are considering underdamped case.

and $\quad R(S) = \frac{1}{s}$

$$\therefore \quad C(S) = \left[\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] \times \frac{1}{S}$$

$$C(S) = \frac{1}{s}\left[\frac{s^2 + 2\xi\omega_n s + \omega_n^2 - (s^2 + 2\xi\omega_n s)}{s^2 + 2\xi\omega_n s + \omega_n^2}\right]$$

$$= \frac{1}{s}\left[1 - \frac{s(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] \qquad = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 - \xi^2\omega_n^2 + \omega_n^2} \qquad = \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \qquad \text{where } \omega_d = \omega_n\sqrt{1 - \xi^2} \text{ rad/sec.}$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \left(\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right) - \left(\frac{\xi\omega_d}{\sqrt{1 - \xi^2}\;(s + \xi\omega_n)^2 + \omega_d^2}\right)$$

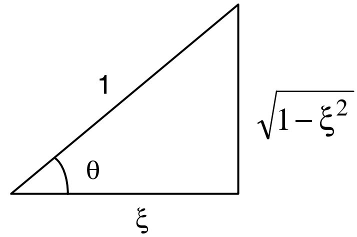
$$\because \quad \omega_d = \omega_n\sqrt{1 - \xi^2}, \; \omega_n = \frac{\omega_d}{\sqrt{1 - \xi^2}}$$

$$C(S) = \frac{1}{s} - \left(\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right) - \left(\frac{\xi}{\sqrt{1 - \xi^2}}\right)\frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}$$

Take inverse Laplace transform,

$$C(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \left(\frac{\xi}{\sqrt{1 - \xi^2}}\right)e^{-\xi\omega_n t}\sin\omega_d t$$

$$\left[\begin{array}{ll} \because L[e^{-at}\sin\omega t = \dfrac{\omega}{(s+a)^2+\omega^2} \\ L[e^{-at}\cos\omega t] = \dfrac{(s+a)}{(s+a)^2+\omega^2} \end{array}\right]$$



$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

$$\cos\theta = \xi, \quad \sin\theta = \sqrt{1-\xi^2}, \quad \tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t]$$

$$\boxed{C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)} \quad \ldots\ldots(A)$$

$[\because \sin(\omega t + \theta) = \sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t]$

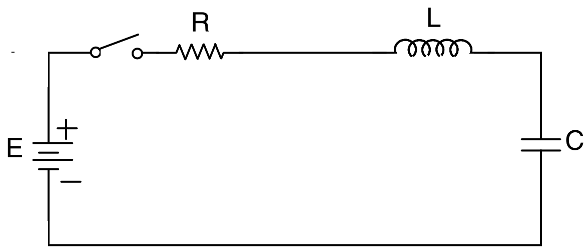
& $\omega_d = \omega_n\sqrt{1-\xi^2}$ rad/sec.

$\theta = \cos^{-1}\xi$ rad

$= \sin^{-1}\sqrt{1-\xi^2}$ rad $= \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$ rad.

APPLICATION OF SECOND ORDER SYSTEM - RLC CIRCUIT

- **For series circuit**



The equation is ,

$$E = L\frac{di}{dt} + Ri + \frac{1}{c}\int_{0^+}^{t} i\,dt$$

Taking Laplace, for unit step

$$\frac{E(s)}{s} = LsI(s) + RI(s) + \frac{1}{Cs}I(s)$$

$$\frac{I(s)}{E(s)} = \frac{1}{s\left(Ls + R + \frac{1}{Cs}\right)} = \frac{\frac{1}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

If we equate the denominator to zero, the characteristic equation is second order.

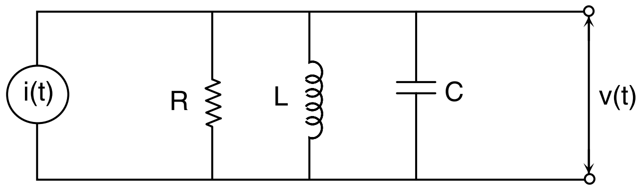
$$i(t) = \frac{ELC}{L}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + \cos^{-1}\zeta\right]$$

$$\omega_n = \frac{1}{\sqrt{LC}}; \qquad 2\zeta\omega_n = \frac{R}{L}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}; \quad \text{steady state } \frac{1}{\omega_n^2} = \frac{\frac{1}{L}}{\frac{1}{LC}} = \frac{LC}{L} = C$$

Natural frequency	$\omega_n = \dfrac{1}{\sqrt{LC}}$
Damping ratio	$\zeta = \dfrac{R}{2}\sqrt{\dfrac{C}{L}} \quad \left(2\zeta\omega_n = \dfrac{R}{L}\right) = \dfrac{1}{2Q}$
Gain	$K = \dfrac{1}{L}$
Damping coefficient	$\alpha = \dfrac{R}{2L}$
Damping natural frequency	$\omega_d = \dfrac{1}{\sqrt{LC}}\sqrt{1-\dfrac{R^2C}{4L}}$
Time constant	$\tau = \dfrac{1}{\alpha} = \dfrac{2L}{R}$

- **For a parallel circuit**



$$i(t) = i(R) + i(L) + i(C)$$

$$= \frac{v(t)}{R} + \frac{1}{L}\int_0^t v(t)dt + C\frac{dv}{dt}$$

Taking Laplace Transform

$$I(s) = \frac{V(s)}{R} + \frac{V(s)}{sL} + sCV(s)$$

$$= V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = V(s)\left(\frac{s^2CRL + sL + R}{sLR}\right)$$

$$G(s) = \frac{V(s)}{I(s)} = Z(s)$$

$$= \frac{sLR}{s^2LCR + sL + R}$$

$$= \frac{sLR}{LCR\left(s^2 + s\dfrac{1}{CR} + \dfrac{1}{LC}\right)} = \frac{s\left(\dfrac{1}{C}\right)}{\left(s^2 + s\dfrac{1}{CR} + \dfrac{1}{LC}\right)}$$

The natural frequency is the same as series circuit $\omega_n = \frac{1}{\sqrt{LC}}$

Damping ratio $\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}} = \frac{1}{2Q}$

Gain $K = \frac{1}{C}$

Damping coefficient $\alpha = \zeta\,\omega_n = \frac{1}{2CR}$

It may be noted in the above derivation that where the input is voltage source and output is current, the transfer function G(s) is an admittance Y(s); where the input is current and output is voltage, the transfer function $\frac{V(s)}{I(s)}$ is impedance Z(s). It may not always be the same.

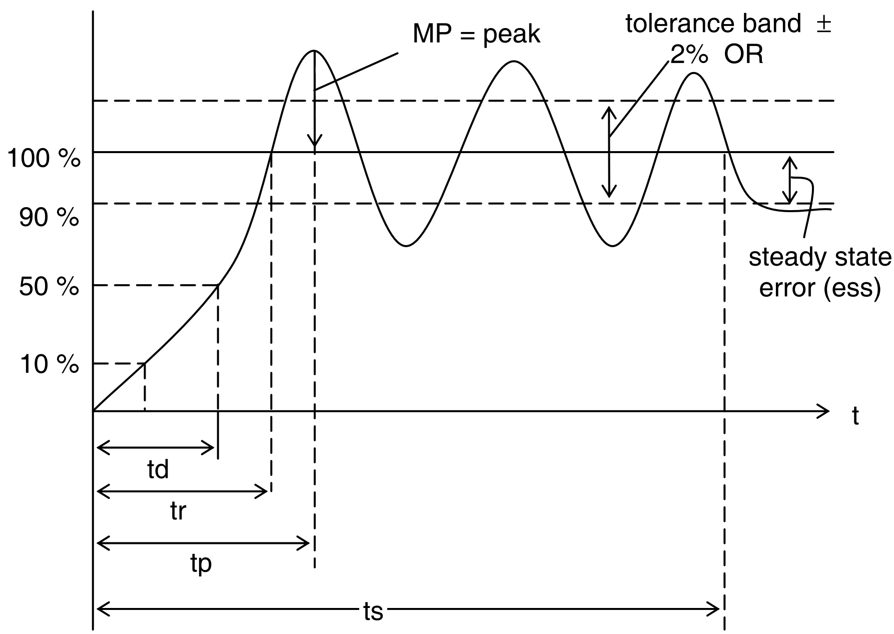
For the series circuit $\frac{V_o(s)}{V_i(s)} = \frac{1}{s^2LC + sCR + 1}$

TYPICAL TIME RESPONSE OF UNDERDAMPED 2ND ORDER SYSTEM AND TRANSIENT RESPONSE SPECIFICATIONS :

The output expression for underdamped 2nd order system is :

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

and it's response is shown in following figure.



Time Response Specifications

1. Delay time (t_d)

It is time required for the response to reach 50% of the final value in first attempt.

$$t_d = \frac{1 + 0.7\ \ \xi}{\omega_n} \text{ second}$$

2. Rise time (t_r)

It is time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped system.

$$t_r = \frac{\pi - \theta}{\omega d} \text{ second where } \theta \text{ must be in radians.}$$

3. Peak time (t_p)

It is time required for response to reach its peak value. (for the first time)

$$t_p = \frac{\pi}{\omega d} \text{ second.}$$

4. Peak overshoot (M_P)

It indicates the normalized difference between the time response peak and steady state output and is given by :

$$M_P\% \quad = C(t)\big|_{t=TP} - 1$$

$$\boxed{M_P\,\% \quad = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100}$$

5. Settling time (T_s)

It is time required for the response to reach and stay_within a specified tolerance band [usually 2% or 5%] of its final value.

$$T_s \; = \; 4\text{ T for 2 \% tolerance band}$$

$$= \; 3\text{ T for 5\% tolerance band where } T = \frac{1}{\xi\omega_n}.$$

6. Steady state error (e_{ss})

It indicates the error between the actual output and desired output as t tends to infinity.

i.e., $$\boxed{e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]}$$

Derivation of T_r, T_p, M_P, T_s & e_{ss}

1. Rise time (T_r)

The rise time tr is obtained when C(t) reaches unity for first attempt i.e.,

$C(t)\,|_{t=tr} = 1$ - For unit step input

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$\because \quad 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_d t_r + \theta] = 1$$

$$= -\frac{e^{-\xi\omega_n t\ r}}{\sqrt{1-\xi^2}} \sin[\omega_d t_r + \theta] = 0$$

$$\sin[\omega_d t_r + \theta] = 0$$
$$[\omega_d t_r + \theta] = \sin^{-1} 0$$
$$[\omega_d t_r + \theta] = n\pi \quad \text{where } n = 1, 2, 3, \ldots..$$

As we are interested in first attempt use n = 1.

$$\omega_d t_r + \theta = \pi$$
$$\omega_d t_r = \pi - \theta$$

$$\boxed{T_r = \frac{\pi - \theta}{\omega d}} \text{ sec. where } \theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}.$$

2. Peak time (t_p)

The time required for the response when it reaches to its peak value.
As $t = T_p$, C(t) will achieve its maxima, according to maxima theorem.

$$\left.\frac{dC(t)}{dt}\right|_{t=tp} = 0$$

So differentiate C(t) w.r.t. time we can write :

$$\left.\frac{dC(t)}{dt}\right|_{t=tp} = 0$$

$$\frac{-e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega d t + \theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d \cos(\omega_d t + \theta)$$

[where $c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$]

Substitute $\omega_d = \omega_n \sqrt{1-\xi^2}$.

$$\frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_d t + \theta] - e^{-\xi\omega_n t} \times \omega_n \cos[\omega_d t + \theta] = 0$$

$$\frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}[\xi \sin[\omega_d \quad t+\theta]-\sqrt{1-\xi^2}\cos[\omega_d t+\theta] \qquad = 0$$

$$\therefore \qquad \xi \sin[\omega_d t+\theta]-\sqrt{1-\xi^2}\cos[\omega_d t+\theta]=0$$

$$\xi \sin[\omega_d t+\theta] = \sqrt{1-\xi^2}\cos(\omega_d t+\theta)$$

$$\frac{\sin[\omega_d t+\theta]}{\cos[\omega_d t+\theta]}=\frac{\sqrt{1-\xi^2}}{\xi}$$

$$\tan[\omega_d t+\theta] = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\tan[\omega_d t+\theta] = \tan\theta \qquad \ldots\ldots(1) \qquad \left[\because \tan\theta=\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

from trigonometric formula :

$$\tan(n\pi+\theta) = \tan\theta \qquad \ldots\ldots(2)$$

equate (1) & (2) : $\omega_d t = n\pi$ where n = 1, 2, 3, ……

But T_p time required for first peak over shoot. $\therefore$ n = 1.

$$\omega_d T_p = \pi$$

$$\boxed{T_p = \frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \text{ second.}$$

3. Peak Overshoot (M_p)

Peak overshoot (M_p) is defined as :

$$M_p = C(t_p) - 1$$

$$M_p = 1-\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin(\omega_d t_p+\theta) - 1$$

$$M_p = -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin[\omega_d t_p+\theta] \qquad \because \quad t_p = \pi/\omega_d$$

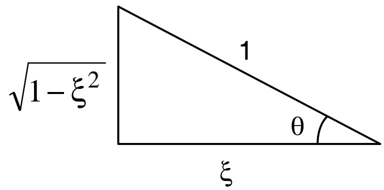
$$M_p = \frac{e^{\frac{-\xi\omega_n\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\sin[(\omega_d\times\frac{\pi}{\omega_d})+\theta]$$

$$= -\frac{e^{\frac{-\xi\omega_n\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\sin[\pi+\theta] \quad [\because \ \omega_d = \omega_n\sqrt{1-\xi^2}\ ]$$

$$M_p = -\frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin[\pi + \theta]$$

Now $\sin[\pi + \theta] = -\sin\theta$

$$M_p = \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}\sin\theta$$



$$\theta = \tan^{-1}x$$

$$x = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\frac{x}{1} = \tan\theta$$

$$\sin\theta = \sqrt{1-\xi^2}$$

$$M_p = \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \times \sqrt{1-\xi^2}$$

$$\boxed{M_p\,\% = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100}$$

4. Settling time (t_s)

Settling time considered only for exponentially decaying envelope for a tolerance band of 2% & 5%.

for 5 % tolerance ts is :

$$e^{-\xi\omega_n t}\Big|_{t=ts} = 0.05$$

$$e^{-\xi\omega_n ts} = 0.05$$

Take log both sides,

$$-\xi\omega_n\, t\, s = \ln 0.05$$

$$\boxed{t_s = \frac{3}{\xi\omega_n}}$$ for 5 % tolerance.

Similarly for 2 % tolerance :

$$\boxed{t_s = \frac{4}{\xi\omega_n}}$$ for 2% tolerance.

5. Steady state error [ess]

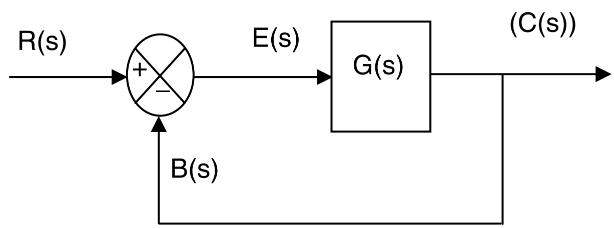
$$e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]$$

$$= \lim_{t\to\infty} \left[1-1+\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin[\omega_d t+\theta]\right] = \lim_{t\to\infty}\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin[\omega_d t+\theta]\right] = 0$$

DETERMINATION OF STEADY STATE ERRORS AND ERROR CONSTANTS

Consider the unity feedback system as below :

R(s) → Input
C(s) → Output
B(s) → Feedback signal
E(s) → Error signal



C.L. Transfer Function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$C(s) = E(s)\,G(s)$$

$$\therefore \quad E(s) = \frac{C(s)}{G(s)} = \frac{R(s)}{1+G(s)}$$

Using final value theorem,

Steady state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s\,R(s)}{1+G(s)} \qquad \text{....(1)}$$

Equation (1) shows that steady state error depend upon the input R(s) and forward transfer function G(s).

Steady State Error for different Input signals :

1. *Unit–step Input* :

Input $r(t) = u(t)$
$R(s) = 1/s$

From equation, $e_{ss} = \lim_{s\to 0}\dfrac{1}{1+G(s)} = \lim_{s\to 0}\dfrac{1}{1+G(0)} = \dfrac{1}{1+K_p}$

where $K_p = G(0)$ is defined as the position error constant.

2. *Unit–ramp Input* :

Input $r(t) = t$ or $\dot{r}(t) = 1$
$R(s) = 1/s^2$

From equation, $e_{ss} = \lim_{s\to 0}\dfrac{1}{s+sG(s)} = \lim_{s\to 0}\dfrac{1}{sG(s)} = \dfrac{1}{K_v}$

where $K_v = \lim_{s\to 0} sG(s)$ is defined as the velocity error constant.

3. *Unit–parabolic (Acceleration) Input* :

Input $r(t) = t^2/2$ or $\ddot{r}(t) = 1$

$R(s) = 1/s^3$

From equation

$$e_{ss} = \lim_{s\to 0} \frac{1}{s^2 + s^2G(s)} = \lim_{s\to 0} \frac{1}{s^2G(s)} = \frac{1}{K_a}$$

where $K_a = \lim_{s\to 0} s^2\, G(s)$ is defined as the acceleration error constant.

(**Note :** K_P, K_v and K_a are defined only for stable systems)

Types of Feedback Control Systems

Open loop transfer function of unity feedback system is

$$G(s) = \frac{k(1+sT_1)(1+sT_2)........}{s^n(1+sT_a)(1+sT_b).......}$$

where k = resultant system gain

n = Type of system

1. Type–0 System

If n = 0, the steady state errors to various inputs, obtained from equations are

$$e_{ss}\,(\text{position}) = \frac{1}{1+G(0)} = \frac{1}{1+K} = \frac{1}{1+K_p}$$

$$e_{ss}\,(\text{velocity}) = \lim_{s\to 0} \frac{1}{sG(s)} = \infty$$

$$e_{ss}(\text{acceleration}) = \lim_{s\to 0} \frac{1}{s^2G(s)} = \infty$$

Thus a system with n = 0 or no integration in G(s) has a constant position error, infinite velocity and acceleration errors. The position error constant is given by the open–loop gain of the transfer function in the time–constant form.

2. Type–1 System

If n = 1, the steady state errors to various inputs, are

$$e_{ss}\,(\text{position}) = \frac{1}{1+G(0)} = \frac{1}{1+\infty} = 0$$

$$e_{ss}\,(\text{velocity}) = \lim_{s\to 0} \frac{1}{sG(s)} = \frac{1}{K} = \frac{1}{K_v}$$

$$e_{ss}(\text{acceleration}) = \lim_{s\to 0} \frac{1}{s^2G(s)} = \infty$$

Thus a system with n = 1 or with one in G(s) has a zero position error, a constant velocity error and an infinite acceleration error at steady–state.

3. Type–2 System

If n = 2, the steady state errors to various inputs are

$$e_{ss}\text{ (position)} = \frac{1}{1+G(0)} = 0$$

$$e_{ss}\text{ (velocity)} = \lim_{s\to 0} \frac{1}{sG(s)} = 0$$

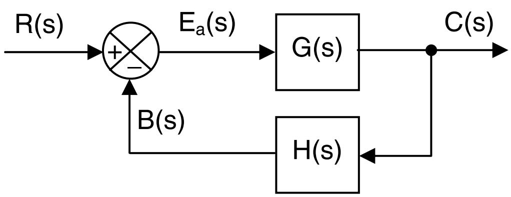
$$e_{ss}\text{(acceleration)} = \lim_{s\to 0} \frac{1}{s^2G(s)} = \frac{1}{K} = \frac{1}{K_a}$$

Thus a system with n = 2 or two integrations in G(s) has a zero position error, zero velocity error and a constant acceleration error at steady state.

- Steady state errors for various inputs and systems are summarized in table below :

Type of input	Steady State Error		
	Type–0 System	Type–1 System	Type–2 System
Unit step	$1/(1 + K_p)$	0	0
Unit–ramp	∞	$1/K_v$	0
Unit–parabolic	∞	∞	$1/K_a$
	$K_p = \lim_{s\to 0} G(s)$	$K_v = \lim_{s\to 0} sG(s)$	$K_a = \lim_{s\to 0} s^2G(s)$

- **For Non–unity Feedback System**



For non–unity feedback systems (figure) the difference between the input signal R(s) and feedback signal B(s) is the actuating error signal ($E_a(s)$) which is given by

$$E_a(s) = \frac{1}{1+G(s)H(s)}R(s)\,.$$

Therefore, the steady–state actuating error is

$$e_{ss} = \lim_{s\to 0} \frac{s\,R(s)}{1+G(s)H(s)}$$

The error constants for non–unity feedback systems may be obtained by replacing G(s) by G(s)H(s) in above table.

Disadvantage of Static Error Coefficient Method

- It cannot give error if inputs are other than the standard test inputs.
- It cannot give precise value of error
- It does not provide variation of error w.r.t. time
- Method is applicable only to stable systems.

- **Dynamic Error Coefficients**

For non–unity feedback systems,

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Let us assume that this is the product of two polynomials of 's'.

$$E(s) = F_1(s) \cdot F_2(s)$$

where $F_1(s) = \dfrac{1}{1+G(s)H(s)}$, $\quad F_2(s) = R(s)$

Now, If $F(s) = F_1(s) \cdot F_2(s)$ then using convolution integral,

$$L^{-1}\{F(s)\} = F(t) = \int_0^t F_1(\tau)F_2(t-\tau)\,d\tau$$

Similarly, $e(t) = \int_0^t F_1(\tau)F_2(t-\tau)\,d\tau = \int_0^t F_1(\tau)\;R(t-\tau)\,d\tau$

$R(t-\tau)$ can be expanded by using Taylor series form as,

$$R(t-\tau) = R(t) - \tau R'(t) + \frac{\tau^2}{2!}R''(t) - \frac{\tau^3}{3!}R'''(t) + ...$$

Substituting $e(t) = \int_0^t F_1(\tau)\left[R(t) - \tau R'(t) + \frac{\tau^2}{2!}R''(t) - \frac{\tau^3}{3!}R'''(t) + ...\right]$

$$= \int_0^t R(t)\,F_1(\tau)\,d\tau - \int_0^t \tau\,R'(t)\,F_1(\tau)\,d\tau + ...$$

Now $\quad e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left[\int_0^t R(t)\,F_1(\tau)\,d\tau - \int_0^t \tau\,R'(t)\,F_1(\tau)\,d\tau + ...\right]$

LIST OF FORMULAE

- **RLC Circuit**
  - Natural frequency $\quad \omega_n = \dfrac{1}{\sqrt{LC}}$
  - Damping ratio $\quad \zeta = \dfrac{R}{2}\sqrt{\dfrac{C}{L}} \;\Rightarrow\; \left(2\zeta\omega_n = \dfrac{R}{L}\right) = \dfrac{1}{2Q}$
  - Gain $\quad K = \dfrac{1}{L}$
  - Damping coefficient $\quad \alpha = \dfrac{R}{2L}$
  - Damping natural frequency $\quad \omega_d = \dfrac{1}{\sqrt{LC}}\sqrt{1-\dfrac{R^2C}{4L}}$
  - Time constant $\quad \tau = \dfrac{1}{\alpha} = \dfrac{2L}{R}$

- **For a parallel circuit**
  - The natural frequency is the same as series circuit $\omega_n = \dfrac{1}{\sqrt{LC}}$
  - Damping ratio $\zeta = \dfrac{1}{2R}\sqrt{\dfrac{L}{C}} = \dfrac{1}{2Q}$
  - Gain $K = \dfrac{1}{C}$
  - Damping coefficient $\alpha = \zeta\,\omega_n = \dfrac{1}{2CR}$

- $\tau = \dfrac{\text{coefficient of time varying term}}{\text{coefficient of steady term}}$ = Time constant

- In the study of control system, linear constant–coefficient second–order differential equations of the form :

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2\, y = \omega_n^2\, u$$

- **Delay time (t_d)**

$$t_d = \frac{1 + 0.7\,\xi}{\omega_n} \text{ second}$$

- **Rise time (t_r)**

$$t_r = \frac{\pi - \theta}{\omega d} \text{ second where } \theta \text{ must be in radians.} \quad \theta = \cos^{-1}\xi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

- **Peak time (t_p)**

$$t_p = \frac{\pi}{\omega d} \text{ second.}$$

- **Peak overshoot (M_P)**

  It indicates the normalized difference

$$M_P\,\% = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

- **Settling time (T_s)**

$$T_s = 4\,T \text{ for 2 \% tolerance band}$$
$$= 3\,T \text{ for 5\% tolerance band where } T = \frac{1}{\xi\omega_n}.$$

- **Steady state error (e_{ss})**

$$e_{ss} = \lim_{t\to\infty}\,[r(t) - c(t)]$$

- **Short Table of Laplace Transform**

Time Function		**Laplace Transform**
Unit Impulse	$\delta(t)$	1
Unit Step	$1(t)$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine wave	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine wave	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Damped Sine wave	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped Cosine wave	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

- **Transient Response Specifications**

1. $C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

   where $\omega_d = \omega_n \sqrt{1-\xi^2}$ and $\theta = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$.

2. delay time (t_d) $= \frac{1+0.7\,\xi}{\omega n}$ second

3. Rise time (t_r) $= \frac{\pi-\theta}{\omega d}$ second

4. Peak time (t_p) $= \frac{\pi}{\omega d}$ second

5. Mp % $= e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$

6. $T_s = 3\,T$ for 5 % tolerance

   $= 4\,T$ for 2 %^ tolerance where $T = \frac{1}{\xi\omega_n}$.

- Steady state errors for various inputs and systems are summarized in table below :

Type of input	Steady State Error		
	Type–0 System	Type–1 System	Type–2 System
Unit step	$1/(1 + K_p)$	0	0
Unit–ramp	∞	$1/K_v$	0
Unit–parabolic	∞	∞	$1/K_a$
	$K_p = \lim_{s\to 0} G(s)$	$K_v = \lim_{s\to 0} sG(s)$	$K_a = \lim_{s\to 0} s^2G(s)$

LMR (LAST MINUTE REVISION)

- The Response of the system as a function of time, to the applied excitation is called Time Response.
- The Free response of a differential equation is the solution of the differential equation when the input u(t) is identically zero.
- The forced response $y_b(t)$ of a differential equation is the solution of the differential equation when all the initial conditions are identically zero.
- The total response of a linear constant –coefficient differential equation is the sum of the free response and the forced response.
- The behaviour of the output from the initial value to the steady state value is called the transient response of the system.
- Note that :
  - If $\zeta > 1$, both poles are negative and real.
  - If $\zeta = 1$, the poles are equal, negative, and real ($s = -\omega_n$)
  - If $0 < \zeta < 1$, the poles are complex conjugates with negative real parts ($s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$)
  - If $\zeta = 0$, the poles are imaginary and complex conjugate ($s = \pm j\omega_n$)
  - If $\zeta < 0$, the poles are in the right half of the s–plane (RHP).

Sr. No.	Range of ξ	Types of closed loop poles	Nature of response	System classification
1	$1 < \xi < \infty$	Real, unequal & negative	Purely exponential	Overdamped
2	$\xi = 1$	Real, equal & negative	Critically pure exponential	Critically damped
3	$0 < \xi < 1$	Complex conjugate with negative real part.	Damped oscillations	Underdamped
4	$\xi = 0$	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped

- The output expression for underdamped 2nd order system is :

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

❑ ❑ ❑ ❑ ❑ ❑

Topic 4 : Stability Analysis

INTRODUCTION

In a system, stability implies small changes in the input do not result in large changes in the output.

- A linear time–invariant system is called to be stable, if the output eventually comes back to its equilibrium after disturbances.
- A linear time invariant system is called as unstable if the output continues to oscillate or increases unboundly from equilibrium state under the influence of disturbance.

> System stable if
> - bounded input results in bounded output
> - zero input makes output tend to zero irrespective of initial conditions.

With reference to the unbounded output, the output of an unstable system extends to certain magnitude. After this the system breakdowns and the linear system is converted to a nonlinear system.

> For nonlinear system, there may or may not be infinite equilibrium states. Hence to define the concept of stability for such multiple existence equilibrium states is very difficult.

If the impulse response of a system is absolutely integrable, i.e.

$\int_0^{\infty} |h(t)\, dt| < \infty$ then the system is said to be stable.

The nature of h(t) depends on poles of the transfer function H(s) which are the roots of characteristic equation.

It reveals :

Roots

- Imaginary $\xrightarrow{\text{poles multiplicity}}$ response increases without bound
- Non zero real part $\xrightarrow{\text{multiplying factor}}$ $e^{\sigma i}$
  - $\sigma < 0$
    - roots have negative real part
    - response vanishes as $t \to \infty$
  - $\sigma > 0$
    - roots have positive real part
    - response increases without bound as $t \to \infty$

Following conclusions can be drawn :

- if roots have negative real part $\rightarrow$ impulse response is bounded. System stable.
- if roots have positive real part $\rightarrow$ system unstable
- if roots are repeated (more than 2) on imaginary axis $\rightarrow$ system is unstable.
- if roots are non repeated (one or more) on imaginary axis $\rightarrow$ system is marginally stable.

  as h(t) is bounded but $\int h(t)\,dt$ is not finite. (oscillatory)

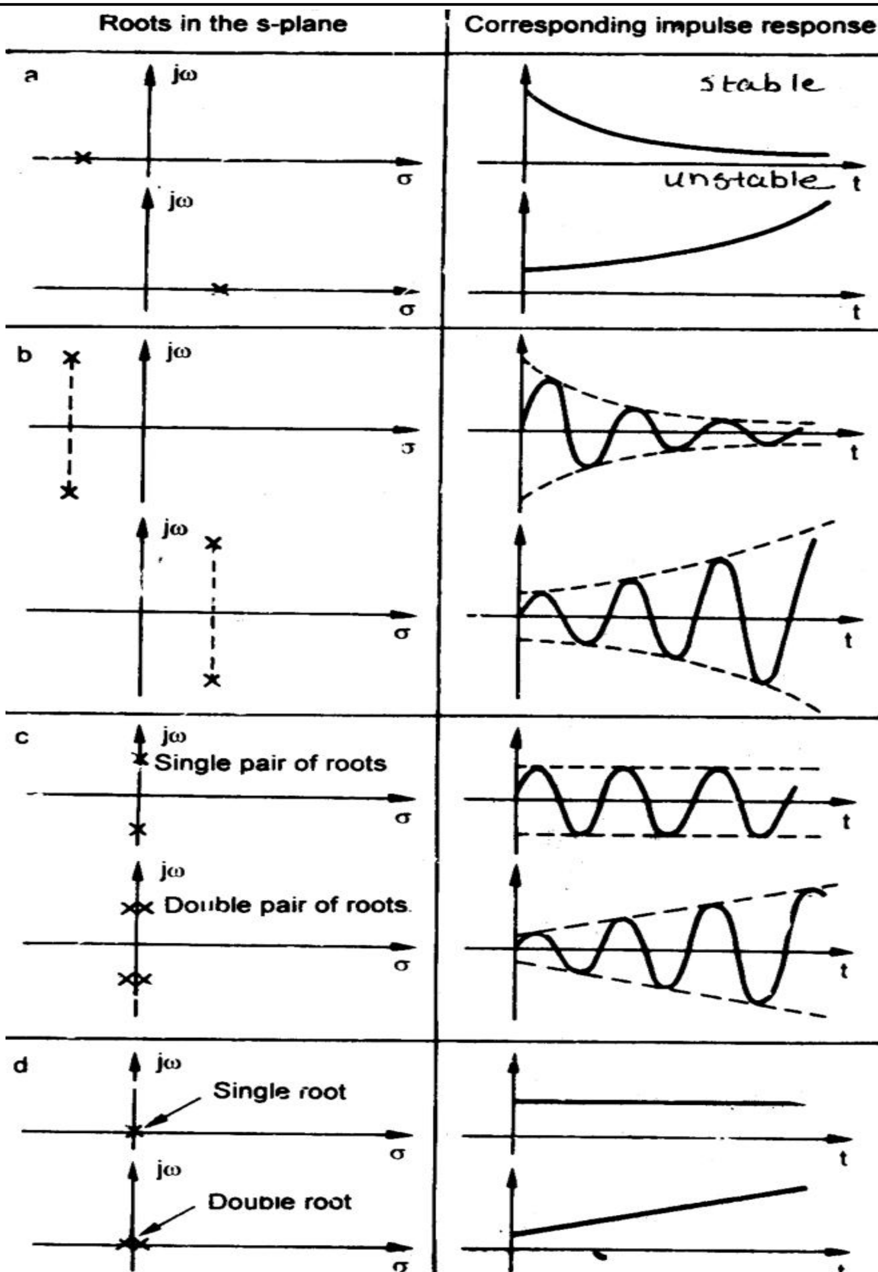
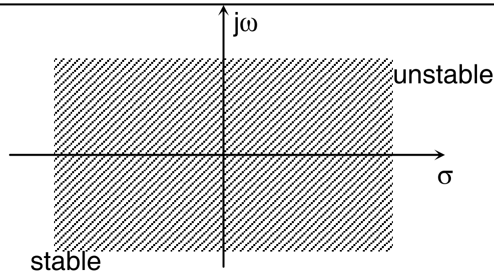


Fig. Response terms contributed by various types of roots.

Closed loop poles in the right half s–plane are not permissible as the system becomes unstable.

Diagrammatically :



Roots have negative real part and also one or more non repeated roots on jω axis then system is limitedly stable.

Stable

Absolutely Stable

- System stable for all the parameters

Conditionally Stable

- System stable for certain range of parameters.

One of the important term is Relative stability. It is a quantity which measures the flow of how fast the transient dies out.

NECESSARY CONDITION FOR STABILITY

The necessary condition for a linear system to be stable is the coefficient of characteristic equation should be real and have same sign and also nonzero.

The coefficients of characteristics equation are positive indicates that roots are real and negative.

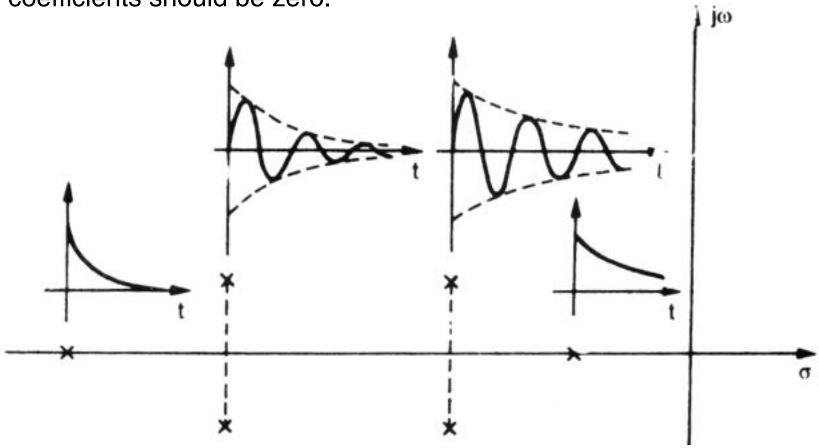
It reveals :

- Positiveness of coefficient is necessary and sufficient condition for stability is valid for first–second order only and not valid for third and higher order.

Necessary Conditions for Stability

i) All the coefficients of characteristics equation q(s) = 0 of the LTI system should be non–zero

ii) None of the coefficients should be zero.



ROUTH'S HURWITZ STABILITY CRITERION

It tells whether or not there are positive roots in polynomial equation without solving them. In other words, this method is used to determine the location of poles of a characteristic equation with respect to the left half and right half of the s–plane without actually solving the equation. The T.F. of any linear closed loop system can be represented as

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{B(s)}{F(s)}$$ where 'a' and 'b' are constants.

To find closed loop poles we equate F(s) = 0. This equation is called as characteristic equation of the system.

i.e. $F(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n$ $\qquad F(s) = 0$

Two methods can be applied :

- **Hurwitz Stability Criterion**
  Consider the characteristic equation as
  $$a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n = 0$$
  According to Hurwitz determinant
  $$\begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots\ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots\ldots & 0 \\ . & . & . & & & \\ . & . & . & & & \\ . & . & . & & & \\ a_{2n-1} & a_{2n-2} & . & . & . & a_n \end{vmatrix}$$

The necessary and sufficient condition for stability is

$\Delta_1 = a_1 > 0$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

and so on…

Note : If $\Delta_{n-1} = 0$ then the system is limitedly stable.

- **Routh's Stability Criterion**

In this method, the coefficients of a characteristic equation are tabulated in a particular way. i.e. it is nothing but ordering the coefficients.

$$F(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0$$

Method of forming an array :

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots. \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots. \\
s^{n-2} & b_1 & b_2 & b_3 & & \\
s^{n-3} & c_1 & c_2 & c_3 & & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^0 & a_n & & & &
\end{array}$$

Coefficients for first 2 rows are written directly from characteristic equation.
From these 2 rows next rows can be obtained as follows :

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

From 2nd and 3rd row, 4th row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n. From this array stability of system can be predicted.

Routh's Criterion

The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in first column of array." If there are any sign changes existing then.

a) System is unstable
b) The number of sign changes equals the number of roots lying in the right half of the s–plane.

- The missing terms in array are regarded as zero.
- All the elements of any row can be divided by positive constant to simplify the computational process.

Example 1:

$$s^3 + 4s^2 + s + 10 = 0$$

Solution **:**

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 10$$

$$\begin{array}{c|cc}
s^3 & 1 & 1 \\
s^2 & 4 & 10 \\
s^1 & \dfrac{4-10}{4} = -\dfrac{6}{4} = -\dfrac{3}{2} & 0 \\
s^0 & 10 &
\end{array}$$

Sign changes = 2

$\therefore$ System is unstable with 2 roots in R.H.S of S plane.

Advantages of Routh's Criterion :

i) Stability of the system can be judged without actually solving the characteristic equation.
ii) No evaluation of determinants, which saves calculation time.
iii) For unstable system it gives number of roots of characteristic equation having positive real part.
iv) Relative stability of the system can be easily judged.
v) By using this criterion, critical value of system gain can be determined hence frequency of sustained oscillations can be determined.
vi) It helps in finding out range of values of K for system stability.
vii) It helps in finding out intersection points of root locus with imaginary axis.

Limitations of Routh's Criterion :

i) It is valid only for real coefficients of the characteristics equation.
ii) It does not provide exact locations of the closed poles in left or right half of s–plane.
iii) It does not suggest methods of stabilizing an unstable system.
iv) Applicable only to linear systems.

Example 2:

Find the range of values of 'k' so that system with following characteristic equation will be stable.

$$F(s) = s\left(s^2 + s + 1\right)(s+4) + k = 0$$

Solution **:**

$$F(s) = s^4 + 5s^3 + 5s^2 + 4s + k = 0$$

s^4	1	5	k
s^3	5	4	0
s^2	4.2	k	0
s^1	$\frac{16.80 - 5k}{4.20}$		0
s^0	k		

For system to be stable there should not be sign change in the first column.

$\therefore$ $k > 0$ from S^0

and $16.8 - 5k > 0$ from S^1

$\therefore$ $16.8 > 5k$

$\therefore$ $3.36 > k$

$\therefore$ $k < 3.36$

$\therefore$ Range of 'k' is

$0 < k < 3.36$

Example 3:

For system $s^4 + 22s^3 + 10s^2 + s + k = 0$ find k_{max} and ω at k_{max}.

Solution **:**

s^4	1	10	k
s^3	22	1	0
s^2	9.95	k	
s^1	$\frac{9.95 - 22k}{9.95}$		0
s^0	k		

Marginal value of 'k' which makes row of s^1 as row of zeros.

$9.95 - 22\, k_{max} = 0$

$k_{max} = 0.4524$

Hence $A(s) = 9.95\, s^2 + k = 0$

$9.95\, s^2 + 0.4524 = 0$

$s^2 = -0.04546$

$s = \pm\, 0.2132$

Hence frequency of oscillations = 0.2132 rad / sec.

Special Cases of Routh's Criterion

Case 1 – First element of any of the rows of Routh's array is zero and same remaining row contains at least one non–zero element.

Effect : The terms in the new row become infinite and Routh's test fails.

e.g. : $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

s^5	1	3	2	
s^4	2	6	1	
s^3	0	1.5	0	Special Case 1
s^2	∞	…	…	Routh's array failed

Following two methods are used to remove above said difficulty.

First Method

Substitute a small positive number 'ε' in place of a zero occurred as a first element in a row. Complete the array with this number 'ε'. Then examine the sign change by taking $\lim\limits_{\varepsilon\to 0}$. Consider above example.

$$
\begin{array}{c|ccc}
s^5 & 1 & 3 & 2 \\
s^4 & 2 & 6 & 1 \\
s^3 & \varepsilon & 1.5 & 0 \\
s^2 & \dfrac{6\varepsilon-3}{\varepsilon} & 1 & 0 \\
s^1 & \dfrac{\dfrac{1.5(6\varepsilon-3)}{\varepsilon}-\varepsilon}{\dfrac{(6\varepsilon-3)}{\varepsilon}} & 0 & \\
s^0 & 1 & &
\end{array}
$$

To examine sign change

$$
\begin{aligned}
\lim_{\varepsilon\to 0} &= \frac{6\varepsilon-3}{\varepsilon} = 6 - \lim_{\varepsilon\to 0}\frac{3}{\varepsilon} \\
&= 6 - \infty \\
&= -\infty \quad \text{sign is negative.}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname*{Lim}_{\varepsilon\to 0}\frac{1.5(6\varepsilon-3)-\varepsilon^2}{6\varepsilon-3} &= \operatorname*{Lim}_{\varepsilon\to 0}\frac{9\varepsilon-4.5-\varepsilon^2}{6\varepsilon-3} \\
&= \frac{0-4.5-0}{0-3} \\
&= +1.5 \quad \text{sign is positive.}
\end{aligned}
$$

Routh's array is,

$$
\begin{array}{c|ccc}
s^5 & 1 & 3 & 2 \\
s^4 & 2 & 6 & 1 \\
s^3 & +\varepsilon & 1.5 & 0 \\
s^2 & -\infty & 1 & 0 \\
s^1 & +1.5 & 0 & 0 \\
s^0 & 1 & 0 & 0
\end{array}
$$

As there are two sign changes, system is unstable.

Second Method

To solve the above difficulty one more method can be used. In this, replace 's' by '1/z' in original equation. Taking L.C.M. rearrange characteristic equation in descending powers of 'z'. Then complete the Routh's array with this new equation in 'z' and examine the stability with this array.

Consider F(s) = $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

Put s = 1/z

$$\therefore \quad \frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

$$z^5 + 2z^4 + 6z^3 + 3z^2 + 2z + 1 = 0$$

$$
\begin{array}{c|ccc}
z^5 & 1 & 6 & 2 \\
z^4 & 2 & 3 & 1 \\
z^3 & 4.5 & 1.5 & 0 \\
z^2 & 2.33 & 1 & 0 \\
z^1 & -0.429 & 0 & \\
z^0 & 1 & &
\end{array}
$$

As there are two sign changes, system is unstable.

Case 2 : If we have rows of zeros.Consider if F(s) = as^5 +ds^4 + bs^3 + es^2 + cs + f = 0 then.

s^5		a	b	c	
s^4		d	e	f	
s^3		0	0	0	←Row of zeros, Special case 2

This indicates nonavailability of coefficient in that row.

A) Procedure to eliminate this difficulty

i) Form an equation by using the coefficients of a row which is just above the row of zeros. Such an equation is called as an Auxillary Equation denoted as A(s). For above case such an equation is

$$A(s) = ds^4 + es^2 + f$$

Note that the coefficients of any row are corresponding to alternate powers of 's' starting from the power indicated against it.

So 'd' is coefficient corresponding to s^4 so first term is ds^4 of A(s). Next coefficient 'e' is corresponding to alternate power of 's' from 4 i.e. s^2 hence the term es^2 and so on.

ii) Take the derivative of an auxillary equation with respect to 's'.

i.e. $$\frac{dA(s)}{ds} = 4ds^3 + 2es$$

iii) Replace row of zeros by the coefficients of $\frac{dA(s)}{ds}$

$$\begin{array}{c|ccc|} s^5 & a & b & c \\ s^4 & d & e & f \\ s^3 & 4d & 2e & 0 \end{array}$$

iv) Complete the array in terms of these new coefficients.

B) Importance of auxillary equation

Auxillary equation is always the part of original characteristic equation. This means the roots of the auxillary equation are some of the roots or original characteristic equation. Not only this but the roots of auxillary equation are the most dominant roots of the original characteristic equation, from the stability point of view.

Note : 1) Roots of auxiliary equation are always symmetrically located.
2) Auxiliary equation is always of even degree.

Example 4:

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Solution:

$$\begin{array}{c|cccc} s^6 & 1 & 8 & 20 & 16 \\ s^5 & 2 & 12 & 16 & 0 \\ s^4 & 2 & 12 & 16 & 0 \\ s^3 & 0 & 0 & 0 & 0 \end{array}$$

Row of zeros

A(s) = $2s^4 + 12s^2 + 16 = 0$

$$\frac{dA}{ds} = 8s^3 + 24s = 0$$

$$\begin{array}{c|cccc}
s^6 & 1 & 8 & 20 & 16 \\
s^5 & 2 & 12 & 16 & 0 \\
s^4 & 2 & 12 & 16 & 0 \\
s^3 & 8 & 24 & 0 & 0 \\
s^2 & 6 & 16 & 0 & \\
s^1 & 2.67 & 0 & & \\
s^0 & 16 & & &
\end{array}$$

No sign change, so system may be stable. But as there is row of zero, system will be (i)marginally stable or (ii) unstable. To examine this solve A(s) = 0.

$$2s^4 + 12s^2 + 16 = 0$$

i.e. $s^4 + 6s^2 + 8 = \left(s^2 + 4\right)\left(s^2 + 2\right) = 0$

This yield imaginary roots at $\pm\sqrt{2i}$ and ± 2i.

∴ System is marginally stable.

Example 5:

Find range of values of 'K' so that system with following characteristic equation will be stable.

F(s) = $s\left(s^2 + s + 1\right)\left(s + 4\right) + K = 0$

Solution:

F(s) = $s\left[s^3 + 5s^2 + 5s + 4\right] + K = 0$

= $s^4 + 5s^3 + 5s^2 + 4s + K = 0$

$$\begin{array}{c|ccc}
s^4 & 1 & 5 & K \\
s^3 & 5 & 4 & 0 \\
s^2 & 4.2 & K & 0 \\
s^1 & \begin{array}{c} 0 \\ \frac{16.8-5K}{4.2} \end{array} & 0 & \\
s^0 & K & &
\end{array}$$

For system to be stable three should not be change in the first column.

∴ K > 0 from s^0 and 16.8 – 5K > 0 from s^1

∴ 16.8 > 5K

∴ 3.36 > K

∴ K < 3.36

∴ Range of 'K' is 0 < K < 3.36

Example 6 :

For unity feedback system $s^4 + 3s^3 + 3s^2 + 2s + K = 0$, determine K_{mar} and ω .

Solution :

$$\begin{array}{c|ccc} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & 2.33 & K & 0 \\ s^1 & \dfrac{4.66-3K}{2.33} & 0 & \\ s^0 & K & & \end{array}$$

$\therefore \quad 4.66 - 3K_{mar} = 0$

$\therefore \quad K_{mar} = 1.555$

$\therefore \quad A(s) = 2.33\,s^2 + K_{mar} = 0 = 2.33\,s^2 + 1.555 = 0$

$s^2 = -0.6667$

$s = \pm\, j\,0.8165$

$\therefore$ Frequency of oscillations = 0.8165 rad/sec.

RELATIVE STABILITY ANALYSIS

We require to know the settling time of the dominant roots so as to calculate the relative stability.

It is inversely proportional to the real part of roots.

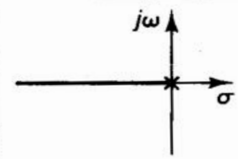
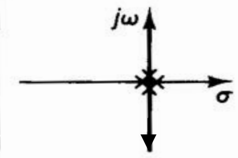
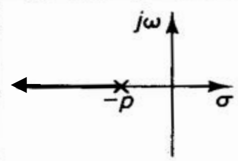
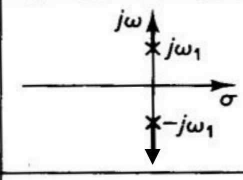
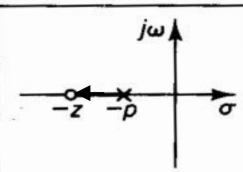
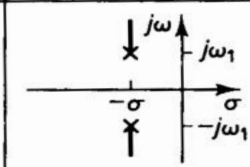
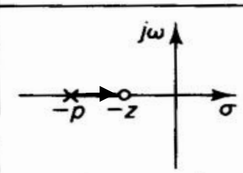
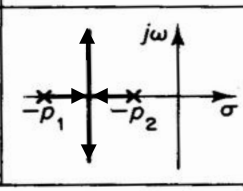
The characteristic equation is modified by shifting the origin of s–plane to s = $-\sigma_1$, by substituting s = z $-\sigma_1$. Now if new equation satisfies Routh criterion, then all roots original equation are more negative than $-\sigma_1$.

ROOT LOCUS

Root locus is the technique employed to find roots of characteristic equation. This technique provides a graphical method of plotting the locus of roots. It brings into focus the dynamic response of the system. The characteristic of transient response of a closed loop system depends upon location of poles. Hence the root locus plays a very vital role in determining the system characteristic.

It provides a measure of sensitivity of roots to variation in the parameters. It is applicable to single as well as multiple loop system.

Collection of simple Root locus plots

$G(s)H(s)$	Open–loop Pole–zero Locations and Root Loci	$G(s)H(s)$	Open–loop Pole–zero Locations and Root Loci
$\frac{K}{s}$		$\frac{K}{s^2}$	
$\frac{K}{s+p}$		$\frac{K}{s^2+\omega_1^2}$	
$\frac{K(s+z)}{s+p}$ $(z > p)$		$\frac{K}{(s+\sigma)^2+\omega_1^2}$	
$\frac{K(s+z)}{s+p}$ $(z < p)$		$\frac{K}{(s+p_1)(s+p_2)}$	

We can define root locus as, the locus of the closed loop poles obtained when the system gain 'k' is varied from $-\infty$ to ∞.

- When k is varied from 0 to ∞, the plot is called as <u>Direct Root Locus</u>.
- When k is varied from $-\infty$ to 0, the plot is called as <u>Inverse Root Locus</u>.

Angle and Magnitude Condition

- *Angle Condition*

  $G(s)\,H(s) = -1 + j0$

  Equating angles of both sides

  $\angle G(s)\,H(s) = \pm (2q + 1)\,180^\circ \qquad q = 0, 1, 2 \ldots$

  $\therefore$ Angle condition can be stated as,

  $\angle G(s)\,H(s)$ for any value of 's' which is root of the equation $1 + G(s)\,H(s) = 0 = \pm (2q + 1)\,180^\circ$, $q = 0, 1, 2, 3\ldots$

> If any point in s–plane has to be on root locus then it has to satisfy above angle condition.

- *Magnitude Condition*

$$\left|G(s)H(s)\right|_{\text{at a point in s-plane which is on root locus}} = 1$$

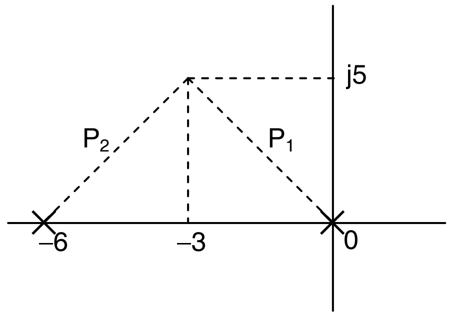
Once a point is known to be on root locus by angle condition, we can use the magnitude condition to find the value of k for which a tested point is one of the roots of the characteristic equation.

Graphical Method of Determining 'k' :

$$k = \frac{\text{Product of phasor lengths drawn from open loop poles upto a point on root locus}}{\text{Product of phasor lengths drawn from open loop}}$$

Example **:**

$$G(s)H(s) = \frac{k}{s(s+6)}$$



Open loop poles are at s = 0, –6

$$P_1 = \sqrt{3^2 + 5^2} = \sqrt{34}$$
$$P_2 = \sqrt{3^2 + 5^2} = \sqrt{34}$$

Now as open loop zeros are absent, denominator is to be assumed unity.

$$k = P_1 \times P_2$$
$$= \sqrt{34} \cdot \sqrt{34} = 34$$

Rules for construction of Root Locus :

Rule No. 1 :

The root locus is always symmetrical about the real axis.

Rule No. 2 :

Let G(s) H(s) = Open loop T.F. of the system

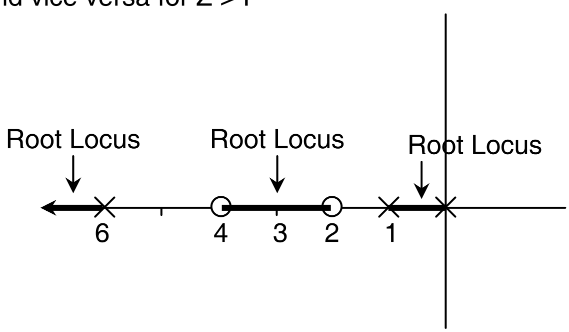
P = Number of open loop poles

Z = Number of open loop zeros

If P > Z, Number of branches N = P and vice versa for Z > P

Rule No. 3 :

A point on real axis lies on the root locus if the sum of the number of open loop poles and the open loop zeros, on the real axis, to the right hand side of this point is odd.



Example:

$$G(s)H(s)=\frac{k(s+2)(s+4)}{s(s+1)(s+6)}$$

Complex conjugate roots should not be considered while using rule number 3

Complex poles must occur in complex conjugate pairs so the root locus is symmetric about real axis.

Rule No. 4 :

Asymptotes are guidelines for the branches approaching to infinity. Angles of such asymptotes are given by,

$$\theta=\frac{\pm(2q+1)180^\circ}{P-Z} \qquad \text{where } q = 0, 1, 2, \ldots. (P - Z - 1)$$

Rule No. 5 :

All the asymptotes intersect the real axis at a common point known as centroid denoted by σ.

$$\sigma=\frac{\sum \text{Real part of poles of } G(s)H(s)-\sum \text{Real parts of zeros of } G(s)\ H(s)}{P\text{ - }Z}$$

Centroid is always real, it may be located on negative or positive real axis. It may or may not be part of root locus.

Rule No. 6 : <u>Breakaway Point</u>

Breakaway point is a point on the root locus where multiple roots of the characteristic equation occurs, for a particular value of K.

The root locus branches always leave breakaway points at an angle of $\pm\frac{180^\circ}{n}$

where n = number of branches approaching at breakaway points.

General Predictions about existence of breakaway points

1. If there are adjacent placed poles on the real axis and the real axis between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed poles.
2. If there are two adjacently placed zeros on real axis and section of real axis in between them is a part of root locus then there exists minimum one break-in point in between adjacently placed zeros.
3. If there is a zero on the real axis and to the left of than zero there is no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one break-in point to the left of than zero.

Determination of Breakaway point :

Step 1 : Construct characteristic equation
$1 + G(s)\,H(s) = 0$

Step 2 : Obtain k in terms of s
$k = f(s)$

Step 3 : Differentiate k w.r.t. 's', equate it zero.

$$\frac{dk}{ds} = 0$$

Step 4 : Roots of equation $\frac{dk}{ds} = 0$ for which k > 0 gives us the breakaway points.

Rule No. 7 :

Intersection of root locus with imaginary axis

Example

$$G(s)H(s) = \frac{k}{s(s+1)(s+4)}$$

Step 1 : $1 + G(s)H(s) = 1 + \frac{k}{s(s+1)(s+4)} = 0$

i.e. $s^3 + 5s^2 + 4s + k = 0$

Step 2 : Routh's Array

s^3	1	4
s^2	5	k
s^1	$\frac{20-k}{5}$	0
s^0	k	

$k_{max} = 20$ that makes row corresponding to s' as row of zeros.

$$\therefore \quad A(s) = 5s^2 + k = 0$$
$$k = k_{max} = 20$$
$$5s^2 + 20 = 0$$
$$s^2 = -4$$
$$\therefore \quad s = \pm j2$$

So $s = \pm j2$ are the points of intersection of root locus with imaginary axis.
If k_{max} is positive there is valid intersection of root locus with imaginary axis.

Rule No. 8 :

The angle at which branch departs from complex pole is called as angle of departure denoted as ϕ_d.

$$\phi_d = 180 - \phi$$

where, $\phi = \Sigma\phi_P - \Sigma\phi_Z$

where, $\Sigma\phi_P$ = Contributions by angles made by remaining poles at the pole at which ϕ_d is to be calculated.

$\Sigma\phi_Z$ = Contributions by the angles made by remaining zeros at the pole at which ϕ_d is to be calculated.

Angle of arrival (ϕ_a) at a complex zero :

$$\phi_a = 180 + \phi$$

where, $\phi = \Sigma\phi_P - \Sigma\phi_Z$

Construction of Root locus :

Determine the portions of the root locus on the real axis. Second calculate the centroid and angles of asymptotes. Draw asymptotes calculate departure and arrival angles at complex poles and zeros. Make a rough sketch of the branches of the root locus so that each branch of locus either terminates at O or approaches ∞ along one asymptotes.

General Steps involved in solving a problem on Root locus :

1. Initially get information about number of open loop poles, zeros, number of branches etc. from G(s) H(s).
2. After the 1^{st} step is over, draw the pole zero plot. Identify sections of real axis for existence of root locus. And predict minimum number of breakaway points by using general predictions.
3. Calculate angles of asymptotes.
4. Determine the centroid.
5. Calculate the breakaway points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as a breakaway points.
6. Calculate the intersection points of root locus with the imaginary axis.
7. Calculate the angles of departures or arrivals if applicable.
8. Combine steps 1 to 7 and draw the final sketch of the root locus.
9. Predict the stability and performance of the given system by using the Root locus.

Gain and phase Margins from Root locus :

It is the factor by which the gain factor k can be multiplied before closed loop system becomes unstable.

$$\text{gain margin} = \frac{\text{value of k at the stability boundary}}{\text{design value of k}}$$

where stability boundary is $j\omega$ axis in s plane or unit circle in z plane. If the root locus does not cross the stability boundary the gain margin is infinite.

- Damping ratio from the root locus :

  Now $\quad GH = \dfrac{k}{(S+P_1)(S+P_2)} \quad\quad k, P_1, P_2 > 0$

  Simply draw a line from origin at an angle of plus or minus θ with negative real axis where

  $$\theta = \cos^{-1}\xi$$

  The gain factor at the point of intersection with root locus is required value of k.

Example:

1. Sketch the complete root locus for the system having

$$G(s)H(s) = \frac{k(s+5)}{(s^2+4s+20)}$$

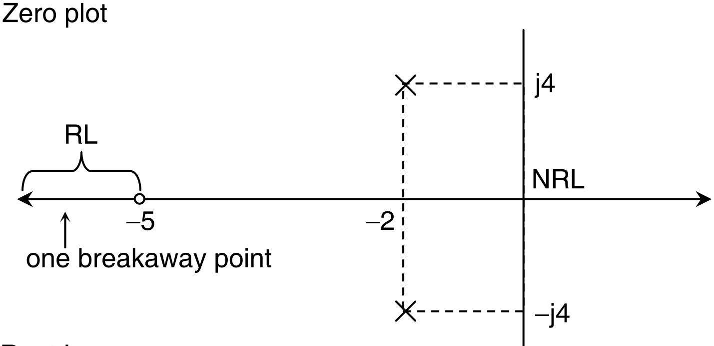
Solution **:**

Step 1 : Number of poles P = 2, Z = 1, N = P . One branch has to terminate at finite zeros s = –5 while P – Z = 1 branch has to terminate at ∞.

Starting points of branches are

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j4$$

Step 2 : Pole – Zero plot



NRL $\rightarrow$ No Root Locus

RL $\rightarrow$ Root Locus

Step 3 : Angles of asymptotes

One branch approaches to ∞ to so one asymptote is required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \quad q = 0$$

$\therefore \quad \theta = 180^\circ$

Branch approaches to ∞ along $+180^\circ$ i.e. negative real axis.

Step 4 : Centroid

As there is only one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway points :

characteristic equation : $1+G(s)H(s)=0$

$$1+\frac{k(s+5)}{(s^2+4s+20)}=0$$

$\therefore \quad s^2+4s+20+ks+5k=0$

$\therefore \quad s^2+4s+20+k(s+5)=0$

$\therefore \quad k = \dfrac{-s^2-4s-20}{(s+5)}$

Now, $\quad \dfrac{dk}{ds} = \dfrac{vu'-uv'}{v^2}=0$

$$= (s+5)(-2s-4)-(-s^2-4s-20)(1)=0$$

i.e. $\quad -s^2 - 10s = 0$

$\therefore \quad -s(s + 10) = 0$

$s = 0$ and $s = -10$ are breakaway points. But $s = 0$ cannot be breakaway point as for $s = 0$, $k = -4$.

For $s = -10$, $\quad k = \dfrac{-100+40-20}{-10+5} = 16$

Hence $s = -10$ is valid breakaway point.

Step 6 : Intersection with imaginary axis characteristic equation.

$$s^2+4s+20+ks+5k=0$$
$$s^2+s(k+4)+(20+5k)=0$$

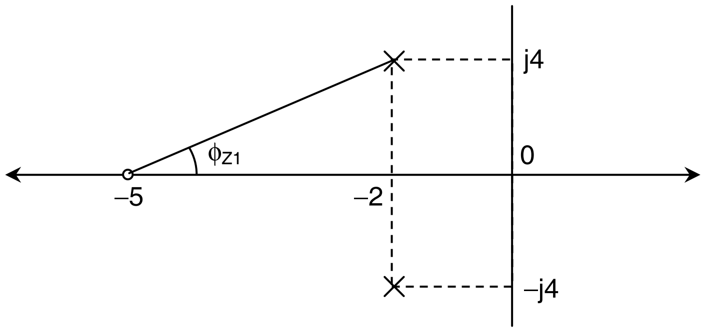
s^2	1	20 + 5k
s^1	k + 4	0
s^0	20 + 5k	

$k_{max} = -4$ makes s row as row of zeros.

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure

Consider –2 + j4 join remaining pole and zero to it.



$$\phi_{P1} = 90^\circ \qquad \phi_{Z1} = \tan^{-1}\frac{4}{3} = 53.13^\circ$$

$$\Sigma\phi_P = 90^\circ \qquad \Sigma\phi_Z = 53.13^\circ$$

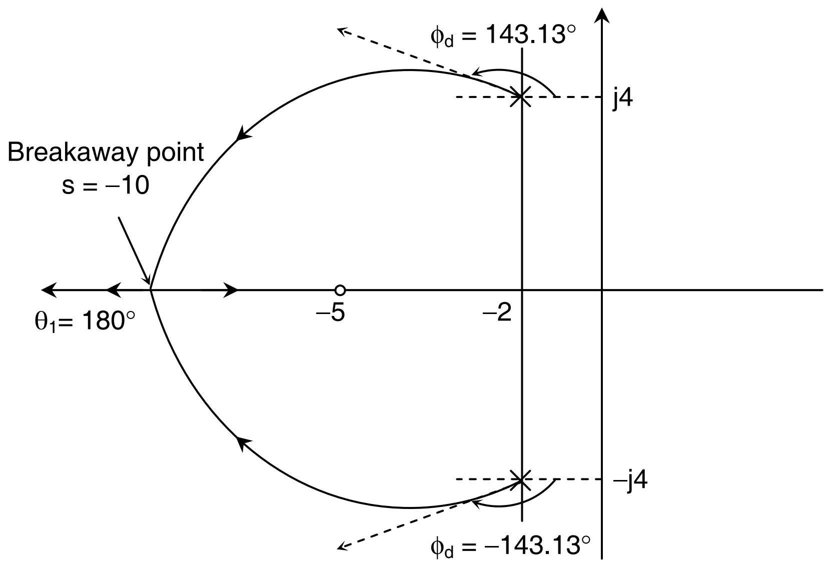
$$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 36.86^\circ$$

$$\phi_d = 180^\circ - \phi$$

$$= 143.13^\circ \quad \text{at} \quad -2 + j4 \text{ pole}$$

$$= -143.13^\circ \text{ at} \quad -2 - j4 \text{ pole}$$

Step 8 : Complete Root Locus is using following figure.



Step 9 : Prediction of Stability

For all ranges of k i.e. 0 < k < ∞, both the roots are always in left half of s–plane. So system is inherently stable.

LIST OF FORMULAE

Rules for Root Locus Construction :

- $\angle G(s)\,H(s)$ for any value of 's' which is root of the equation
  1 + G(s) H(s) = 0 = ± (2q + 1) 180°
  q = 0, 1, 2, 3

  If any point in s–plane has to be on root locus then it has to satisfy above angle condition.

- $\left|G(s)H(s)\right|_{\text{as a point in s-plane which is on root locus}} = 1$

  Once a point is known to be on root locus by angle condition, we can use the magnitude condition to find the value of k for which a tested point is one of the roots of the characteristic equation.

- Asymptotes are guidelines for the branches approaching to infinity. Angles of such asymptotes are given by,

  $$\theta = \frac{(2q+1)180^\circ}{P-Z} \qquad \text{where } q = 0, 1, 2, \ldots (P - Z - 1)$$

- All the asymptotes intersect the real axis at a common point known as centroid denoted by σ.

  $$\sigma = \frac{\sum \text{Real part of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)\ H(s)}{P - Z}$$

- The angle at which branch departs from complex pole is called as angle of departure denoted as ϕ_d.

  $\phi_d = 180 - \phi$

  where, $\phi = \Sigma\phi_P - \Sigma\phi_Z$

  where, $\Sigma\phi_P$ = Contributions by angles made by remaining poles at the pole at which ϕ_d is to be calculated.

  $\Sigma\phi_Z$ = Contributions by the angles made by remaining zeros at the pole at which ϕ_d is to be calculated.

- Angle of arrival (ϕ_a) at a complex zero :

  $\phi_a = 180 + \phi$

  where, $\phi = \Sigma\phi_P - \Sigma\phi_Z$

LMR (LAST MINUTE REVISION)

- A linear time–invariant system is called to be stable, if the output eventually comes back to its equilibrium.

- A linear time invariant system is called as unstable if the output continues to oscillate or increases unboundly from equilibrium state under the influence of disturbance.

- If the impulse response of a system is absolutely integrable,

  i.e. $\int_0^\infty |h(t)\,dt| < \infty$ then the system is said to be stable.

  Following conclusions can be drawn :

  – If roots have negative real part $\rightarrow$ impulse response is bounded. System stable.
  – If roots have positive real part $\rightarrow$ system unstable
  – If roots are repeated (more than 2) on imaginary axis $\rightarrow$ system is unstable.
  – If roots are simple but non repeated (one or more) on imaginary axis $\rightarrow$ system is marginary stable as h(t) is bounded but $\int h(t)\,dt$ is not finite, output is oscillatory.

- Closed loop poles in the right half s–plane are not permissible as the system becomes unstable

- Roots have negative real part and also one or more non repeated roots on $j\omega$ axis then system is limitedly stable.

- The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in first column of array." If there are any sign changes existing then.
  a) System is unstable
  b) The number of sign changes equals the number of roots lying in the right half of the s–plane.

- Routh's Criterion is valid only for real coefficients of the characteristic equation.

- Centroid is always real, it may be located on negative or positive real axis. It may or may not be part of root locus.

- If there are adjacent placed poles on the real axis and the real axis between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed poles.

- If there are two adjacently placed poles or zeros on real axis and section of real axis in between them is a part of root locus then there exists minimum one breakaway point in between adjacently placed poles or zeros.

- If there is a zero on the real axis and to the left of than zero there is no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one breakaway point to the left of than zero.

- Root locus is the technique employed to find roots of characteristic equation. This technique provides a graphical method of plotting the locus of roots. It brings into focus the dynamic response of the system.

- If any point in s–plane has to be on root locus then it has to satisfy $\angle G(s)\,H(s)$ for any value of 's' which is root of the equation $1 + G(s)\,H(s) = 0 = \pm(2q+1)\,180^\circ$, $q = 0, 1, 2, 3 \ldots$

- Gain margin = $\dfrac{\text{value of k at the stability boundary}}{\text{design value of k}}$

❑ ❑ ❑ ❑ ❑ ❑

Topic 5 : Frequency Response Analysis

INTRODUCTION

The steady state response of a system to be sinusoidal input is called as Frequency response. The magnitude and phase relationship between sinusoidal input and steady state output is frequency response.

> In LTI system, the frequency response is independent of amplitude and phase of input and also initial conditions.

Frequency response can be used to get the necessary information for computation of transfer function. It also provides the ease and accuracy of measurement. Effect of undesirable noise can be eliminated in the system designed using frequency response.

> Frequency response analysis can be extended to non–linear systems.

There is an indirect correlation between frequency response and transient response we adjust the frequency response of a system to get desired transient response.

> Routh criterion is a time domain approach to check the stability condition. Root locus is a powerful time domain method for stability.

Nyquist criterion is powerful frequency domain method for stability check of a system. Frequency response can directly be obtained from transfer function simplify by substituting ‘s’ by ‘jω’. Absolute and relative stability can both be found easily with frequency domain approach. The apparatus used for obtaining frequency response is simple, inexpensive and easy to use.

TYPES OF SPECIFICATION

Types

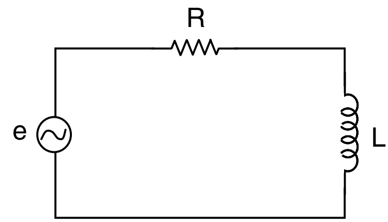
Time Domain specification
- Delay time
- Peak time
- Settling time
- Peak Oversheet
- Rise Time

Frequency Domain specification
- Gain Margin
- Phase Margin
- Delay time
- Bandwidth
- Cut off Rate
- Resonant Peak
- Resonant Frequency

Note : We are concerned with frequency domain.

For example :

Consider RL circuit as shown in figure.



- In time domain analysis :

$$e = Ri + L\frac{di}{dt}$$

Laplace transform :

$$E(s) = RI(s) + LsI(s)$$

$$\therefore \quad I(s) = \frac{E(s)}{(R + sL)}$$

- In frequency domain analysis :
we get

$$I(j\omega) = \frac{E(j\omega)}{R + j\omega L}$$

Consider second order system :
General Transfer function is given as,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad \text{....(1)}$$

ω_n : natural frequency, ξ : damping ratio.

In frequency domain analysis, substitute s = $j\omega$

$$\therefore \quad T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} \qquad \text{....(2)}$$

$$= \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$T(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j(2\xi\omega/\omega_n)}$$

$$T(j\omega) = \frac{1}{\left(\frac{\omega_n^2 - \omega^2}{\omega_n^2}\right) + j\left(2\xi\omega/\omega_n\right)}$$

$$T(j\omega) = \frac{1}{\left(\frac{\omega_n^2 - \omega^2}{\omega_n^2}\right) + j\left(2\xi\frac{\omega}{\omega_n}\right)}$$

$$|T(j\omega)| = M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \text{... (3)}$$

The frequency at which M has a peak value is termed as resonant frequency. The slope of magnitude curve is zero. The frequency is termed as ω_r i.e. 'ω' is replaced by 'ω_r'.

The maximum value at this frequency is termed as Resonant peak (M_r).

Taking the derivative of equation (3) w.r.t. $\left(\frac{\omega_r}{\omega_n}\right)$ to get peak value, we get

$$\omega_r = \omega_n \sqrt{1-2\xi^2} \qquad \text{....(4)}$$

Again substituting equation (4) we get Resonant peak.

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \qquad \text{....(5)}$$

Equation (4), (5) reveals,

- As $\xi \to 0$, $\omega_r \to \omega_n$, $M_r \to \infty$
- As $0 < \xi < 0.707$, $\omega_r < \omega_n$ always, $M_r > 1$
- As $\xi > 0.707$, magnitude of M decreases monotonically from 1 to 0.

For satisfactory operation, the range of ξ is generally $0.4 < \xi < 0.707$

Typical magnitude curve of Feedback Control System

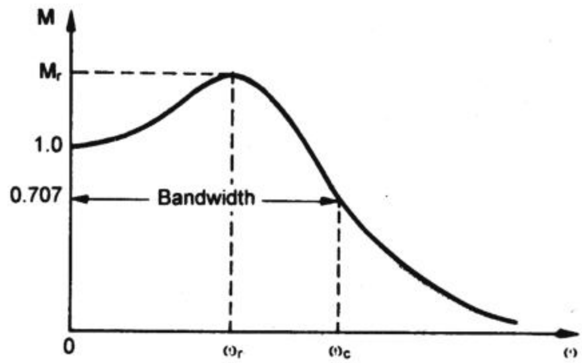


Fig. Typical magnification curve of a feedback control system

The frequency at which value of M is $\frac{1}{\sqrt{2}}$ is called as cut off frequency (ω_c). The range of frequencies over which M is equal to or greater than $\frac{1}{\sqrt{2}}$ is defined as bandwidth (ω_b).

Note :

Bandwidth is equal to the cut off frequency. It indicates the noise filtering characteristic of the system.

To find the normalized bandwidth :

$$M = \frac{1}{\sqrt{\left(1-\frac{\omega_b^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega_b}{\omega_n}\right)^2}} = \frac{1}{\sqrt{2}} \qquad \ldots(6)$$

Solving from equation (6) to get ω_b

$$\frac{\omega_b}{\omega_n} = \left[1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$$

The above expression gives normalized bandwidth.
The denormalized bandwidth is given by

$$\omega_b = \omega_n\left[1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$$

According to the time response, the peak overshoot (M_p) and resonant peak (M_r) both are function of ξ. The correlation is as shown below.

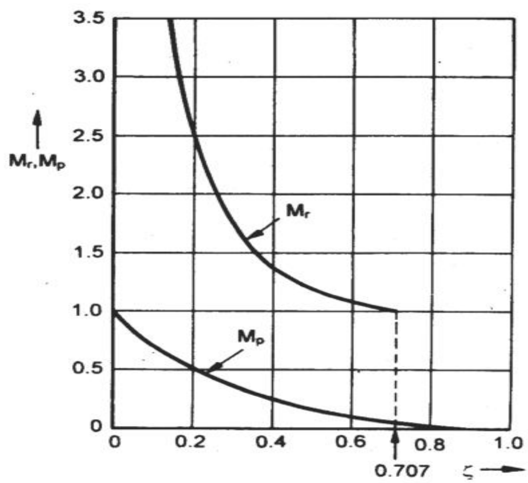
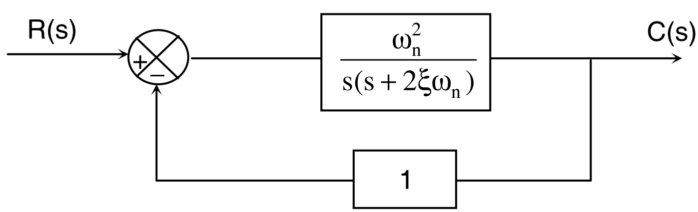


Fig. M_r, M_p versus ζ

Note : For ξ > 0.707, M_r does not exist and correlation between M_r and M_p is not possible.

CORRELATION BETWEEN TRANSIENT RESPONSE AND FREQUENCY RESPONSE

Consider a system as shown below



For unit step system, the output of system is given by

$$C(t) = 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

Now $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

The magnitude of G(jω) becomes unity when

$$\omega = \omega_n\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}$$

Phase margin γ is

$$\gamma = 180 + \angle G(j\omega)$$

$$= 90 - \tan^{-1}\left(\frac{\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}}{2\xi}\right)$$

$$= \tan^{-1}\frac{2\xi}{\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}}$$

Note: Phase margin and damping ratio are correlated.

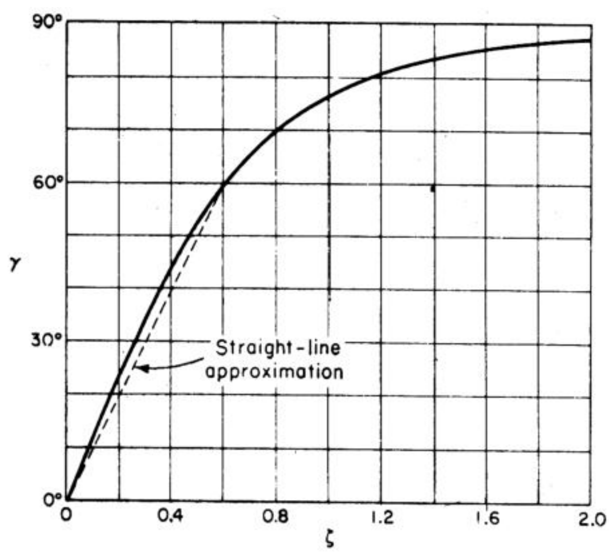


Fig. Curve γ (phase margin) versus ζ for the system shown in fig

The relation between phase margin and damping ratio (for linear region i.e. $\gamma \le 60°$)

$$\xi = \frac{\gamma}{100}$$

Thus phase margin of 60° corresponds to damping ratio of 0.6.

FREQUENCY RESPONSE

Singularities of the function F(s)

There are some points in the s–plane for which F(s) will be infinite. These points in the ‘s’ plane are called the poles of F(s). Poles are singularities of F(s).

Analytic Function F(s)

A function F(s) is said to be analytic at a given value of s provided F(s) and all its derivatives with respect to s exist. The points where F(s) or its derivatives do not exist are called singularities of F(s). As pointed out earlier at certain value of s, the function F(s) is infinite and these values of s are called poles. Poles are singularities of F(s).

A function F(s) is said to be analytic (i.e., F(s) and its derivative exist) except at the singular points.

Single valued and Multi valued complex function

A function such as F(s) = $\sqrt{s}$ has two values for each value of ‘s’ and is said to be multivalued function. One is to one mapping from s plane contour (path) is not possible into F(s) plane contour (path) in such cases.

Principle of argument

The angle of F(s) is also called ‘argument’ of F(s) and generally written as ARG [F(s)]. A theorem regarding mapping of single valued functions which are analytic at all points in ‘s’ plane except a finite number of singularities, is known as the principle of Argument.

According to this principle if the s plane contour contain Z zeros and P poles of F(s) within it, then the mapped F(s) plane contour encircles the origin (Z – P) times in the same direction as the s plane contour. The total change in the angle of F(s) = 2π (Z – P) for a closed contour in the s plane within which there are Z zeros and P poles of F(s).

Application of the principle of argument of stability

Since the closed loop transfer function is given by

$$(s) = \frac{G(s)}{1+G(s)H(s)}$$

Suppose we draw a contour r in the s plane which has the whole of the R.H.S. of s–plane within it and map the function 1 + G(s) H(s) corresponding this contour then the number of enrichment of origin by the mapped contour N will be given by

$$N = Z - P \quad \text{or} \quad Z = N + P$$

Then for stability ‘Z’ must be zero. Hence for stability the contour of 1 + G(s) H(s) corresponding to r must go around the origin – P times (opposite to the direction of r contour).

P is number of poles of [1 + G(s) H(s)]. At any ‘s’ if G(s) H(s) is infinite 1 + G(s) H(s) is also infinite. Thus ‘P’ is also the number of poles of open loop transfer function G(s) H(s).

The contour r is know as **Nyquist Path** (contour) and must be so drawn that the whole of R.H.S. of s–plane is within it.

The figure shows a typical Nyquist path and corresponding mapped 1 + G(s)H(s) contour. Here N = 2. This system cannot represent a stable system. This is because Z = N + P, if N = 2 and P is a +ve number Z cannot be zero.

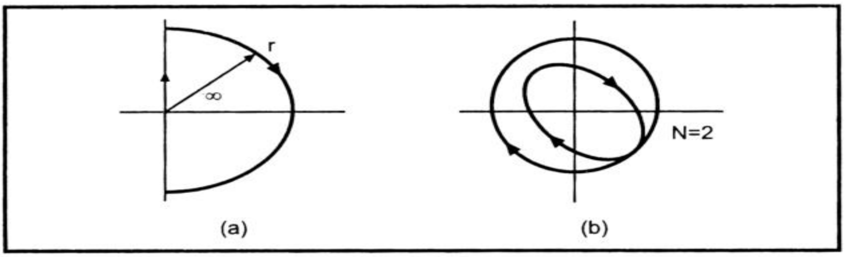


Fig. (a) Nyquist path (r) (b) 1 + G(s) H(s) contour corresponding to r

We can make the matter more simple by saying that G(s) H(s) contour should encircle (–1, 0) point N times in the contour clockwise direction for a stable system where N = –P and where P is the number of open loop poles within the RHP of s–plane.

NYQUIST STABILITY CRITERION

From the knowledge of open loop frequency response, relative and absolute stability can be determined. To investigate both relative and absolute stability, Nyquist stability criterion is used. It relates the location of roots of characteristic equation to open–loop frequency response of the system.

The two main characteristics of frequency domain system are gain and phase.

According to the stability criterion,
In a closed loop transfer function, the denominator is equated to zero.

$$1 + G(s)\,H(s) = 0$$

The nature of the roots of this equation determines absolute stability. If roots are on left half of s–plane then the system is stable.

Note: There is no need to compute the closed loop poles. The stability can be determined by graphical analysis of open loop transfer function.

Nyquist Contour

Nyquist Theorem for Stability

Open loop stable systems P = 0 :

When G(s) H(s) has no poles in the R.H.S., the encirclement of G(s) H(s) contour corresponding to Nyquist path around the critical point (–1, 0) must be zero. i.e., the G(s) H(s) contour should not encircle the critical point. Only then the closed loop system is stable.

Open loop unstable system (P ≠ 0) : If G(s) H(s) has P poles in the R.H.S. of s plane then the G(s) H(s) contour corresponding to the Nyquist path should go round the critical point (–1, 0) P times and in the direction opposite to the Nyquist path, for a stable system. Actually Z = N + P represents the closed loop poles in the R.H.S. of s plane and if Z does not come out to be zero then system is Unstable.

Note that Z cannot be negative. If we get a negative answer we should suspect that we have made some mistake in mapping.

For a given continuous closed path in s–plane, which does not pass through singular point will map a curve in F(s) plane.
Suppose the polynomial be

$$F(s) = \frac{k(s+z_1)(s+z_2)(s+z_3)......(s+z_m)}{(s+P_1)(s+P_2)..........(s+P_n)}$$

for each value of 's', there will be corresponding value of F(s) i.e. $F(s_1)$, $F(s_2)$, ...$F(s_n)$
Now we can transform s_0, s_1, s_2,..... in s–plane to $F(s_0)$, $F(s_1)$..... in F(s) plane.
The number and direction of encirclements of origin of F(s) plane by closed curve is very important to determine stability.

The contour in F(s) plane is called as Nyquist plot.

STABILITY ANALYSIS

Consider a closed contour in the right half s–plane. The contour consists of entire jω axis and ω extends from –∞ to +∞. Such a contour is called as Nyquist path or Nyquist contour.

The contour encloses all zeros and poles of 1 + G(s)H(s) that have positive real parts. If G(s)H(s) has pole or poles at origin of s–plane, mapping of point s = 0 in the F(s) plane becomes indeterminate.

Note that 1 + G(jω) H(jω) is the vector sum of unit vector and vector G(jω) H(jω).

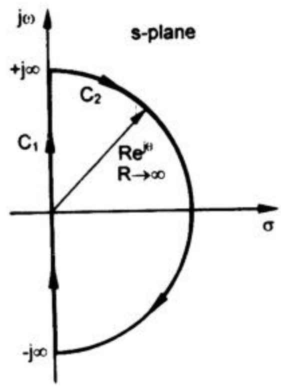


Fig. The Nyquist Contour

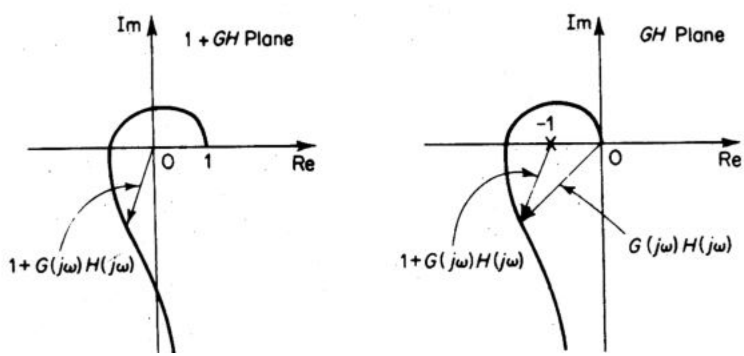


Fig. Plots of 1 + G(jω) H(jω) in the 1 + GH plane and GH plane

1 + G(jω) H(jω) is identical to vector drawn from −1 + j0 to terminal point of vector G(jω) H(jω).

Encirclement of origin by graph 1 + G(jω) H(jω) is equivalent to encirclement of point −1 + j0 by G(jω) H(jω). The clockwise rotations can be counted as we move from $\omega = -\infty$ to $\omega = 0$ to $\omega = \infty$.

These are the basis for Nyquist stability criterion.

Stability Criterion

Statement : If

- open loop transfer function has 'n' poles in right half of s−plane
- also $\lim\limits_{s \to \infty} G(s)\,H(s)$ = constant

then for stability

- locus of g(jω) H(jω) must encircle −1 + 0j 'n' times
- Encirclement should be in counterclockwise direction as ω varies from −∞ to +∞.

Mathematically,

Let q(s) = 1 + G(s) H(s) have z zeros and p poles in the right half s−plane.
The criterion may be expressed as

$$N = P - Z$$

where N : number of counter clockwise encirclements of point −1 + j0
P : number of poles of G(s) H(s) in right half
Z : number of zeros of 1 + G(s) H(s) in the right half s−plane.

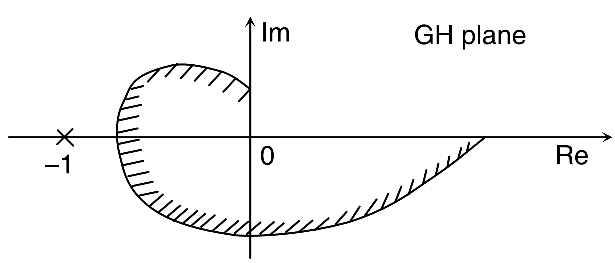
For stable system : There should be no zeros in the right half s−plane.

∴ $z = 0$

∴ $z = 0$ is possible if and only if $N = p$

i.e. number of counter clockwise encirclements of −1 + j0 point = number of poles in the right half

If G(s) H(s) does not any poles in the right half s−plane then Z = N. Thus there should be no encirclements of −1 + j0 point. The stability can be checked by seeing whether −1 + j0 points lie outside the shaded region or not. For stability −1 + j0 point must lie outside as shown in figure.



For stability of multiple pole system :

Encirclement of −1 + j0 point is not sufficient to check whether multiple pole system is stable or not. In such cases, Routh Hurwitz stability criterion can be applied to detect whether poles are on right half of s−plane or not.

If transcendental functions such as e^{-Ts} is included in Transfer function of G(s) H(s). Then before application of Routh stability criterion, the exponential series should be truncated by Taylor series

$$e^{-Ts} \cong \frac{1-\frac{T_s}{2}}{1+\frac{T_s}{2}}$$

Note : If locus of G(jω) H(jω) passes through −1 + j0 then zeros of characteristic equation lies on imaginary axis which is not desirable in practical cases.

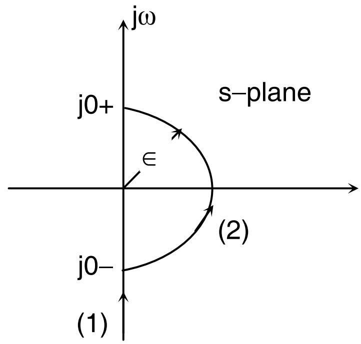
<u>Special Case</u> :

Suppose G(s) H(s) has poles and zeros on jω axis,

Note : Nyquist path should not pass through poles or zeros of G(s) H(s).

We consider G(s) H(s) has a pole or zero at origin (or any other location on jω axis). In this case we use a semicircle of radius infinitesimally small $\in$ and then move a representative point 's' along the contour.

The path of 's' is as shown. As $\in \to 0$ the entire right half plane is covered. Thus it encloses entire poles and zeros on the right half plane.



- To examine the stability of control systems using Nyquist stability criterion, the following possibilities must be checked for :
  a) There is no encirclement of −1 + j0 point. This implies that the system is stable if there are no poles of G(s) H(s) in RHS of s–plane; otherwise the system is unstable.
  b) There is a counterclockwise encirclement or encirclements of −1+j0 point. In this case, system is stable if the number of counterclockwise encirclements is the same as the number of poles of G(s) H(s) in RHS of s–plane; otherwise the system is unstable.
  c) There is a clockwise encirclement or encirclements of −1 + j0 point. In this case system is unstable.

Note : On semicircular path with radius $\in$, complex variable 's' can be written as $s = \in e^{j\theta}$. θ varies from –90 to +90.

Suppose

$$\therefore \qquad G\left(\in e^{j\theta}\right)\ H\left(\in e^{j\theta}\right) = \frac{K}{\in e^{j\theta}\left(T \in e^{j\theta} + 1\right)} \cong \frac{K}{\in e^{j\theta}} = \frac{K}{\in} e^{-j\theta}$$

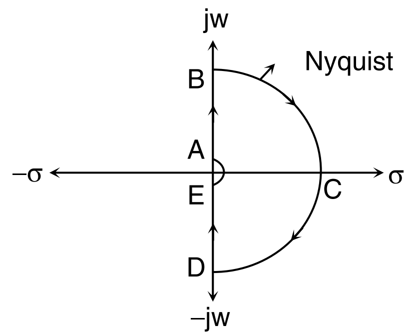
As $\in \to 0, \ \frac{K}{\in} \to \infty$

Thus point $G(j0^-)\ H(j0^-) = j\infty$ and $G(j0^+)\ H(j0^+) = -\ j\infty$
These points are joined by semicircle of radius ∞ on right half plane.

Consider the following example :

- Discuss the stability of system with open loop transfer function $G(s)H(s) = \frac{K}{s^2(s+5)}$ using Nyquist plot.

Solution :



a) Section AB :

$s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K}{\left(\sqrt{\omega^2+5^2}\right)\omega^2}$$

$$\phi = -180^\circ - \tan^{-1}\left(\frac{\omega}{a}\right)$$

Thus, it can be seen that angle varies from -180° to -270° and $G(j\omega)\ H(j\omega)$ continuously decreases with increasing ω.

b) Section BCD :
This section maps into a point at origin.

c) Section DE :
Here, $s = -j\ \omega$. Hence result is a mirror image of $A'B'$

d) Section EFA :

$$G(s)H(s) = \frac{K}{(\varepsilon e^{j\phi})^2(\varepsilon e^{j\phi}+5)} \quad \varepsilon \to 0$$

If we neglect $\varepsilon e^{j\phi}$ component to 5

$$G(s)H(s) = \frac{K}{\varepsilon^2} e^{-j2\phi}$$

As we can see that when Nyquist path undergoes rotation by 180° in counter clockwise direction, G(s)H(s) undergoes rotation by double the angle in clockwise direction with infinite radius.

e) Comment on Stability :

In this case there is no pole of G(s)H(s) in Nyquist path

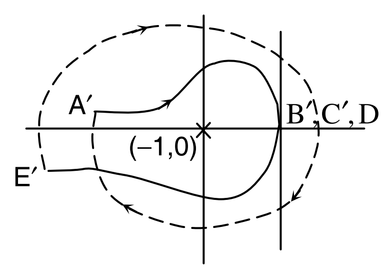
$\therefore \quad N = 2$

We know that,

$$\cancel{Z} = N + P$$

$$\therefore \quad \cancel{Z} = 2 + 0$$

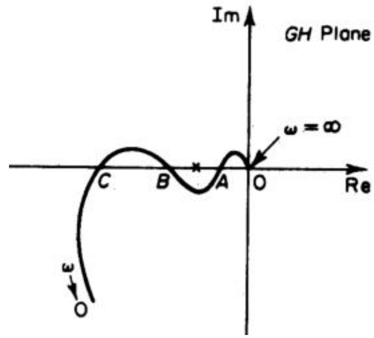
$$\therefore \quad \cancel{Z} = 2$$



But $\cancel{Z}$ must be zero for the system to be stable.
Thus, the system is unstable for all K and has two RHS poles in closed loop.

CONDITIONAL STABILITY SYSTEM

A system is said to have conditional stability if the system is stable for some value of open loop gain for which −1 + j0 is completely outside locus and lying between critical values only. Such a system is unstable if gain is increased or decreased sufficiently.



RELATIVE STABILITY OF A SYSTEM

Nyquist plot helps to determine both absolute stability as well as relative stability of a feedback system.

- **Measure of relative stability**

  The stability information can be gathered by detecting the encirclement of the point −1 + j0. As polar plot gets closer to (−1 + j0) point, the system tends towards instability. Thus system A is more stable than system B.

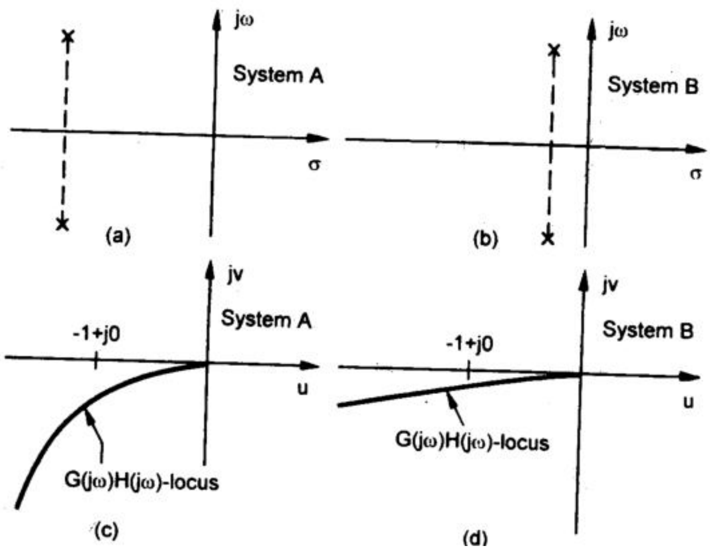
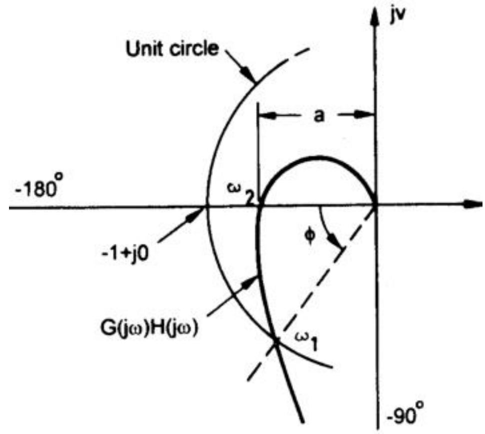


Fig. Correlation between the closed−loop s−plane root locations and open−loop frequency response curve

Consider the locus as shown below

Note, as the locus of G(jω) H(jω) approaches −1 + 0j the system becomes relatively unstable. Also as 'a' approaches 1, φ approaches zero. Thus 'a' and 'φ' are the parameters used to determine relative stability. The practical measures of relative stability evolved from this concept are Phase margin and Gain Margin.

To represent the closeness of locus of G(jω) H(jω) with the point −1 + j0, phase margin and gain margin is used.



PHASE MARGIN AND GAIN MARGIN

Phase Margin

It is the amount of additional phase lag at gain crossover frequency required to bring the system on verge of instability.

Gain cross–over frequency

It is the frequency at which open–loop transfer is unity.

Note :

- Phase margin and gain margin both should be given to determine the relative stability.
- For minimum phase system, both phase and gain margin should be positive. Negative margin indicates instability.
- For satisfactory performance, the phase margin should be between 30° and 60° and gain margin should be greater than 6 dB.

POLAR PLOT

Polar plot is the plot of magnitude of sinusoidal transfer function versus the phase angle. Polar plot is obtained as ω is varied from zero to infinity. Positive phase angle is measured counterclockwise from positive real axis. The polar plot is often called as Nyquist plot.

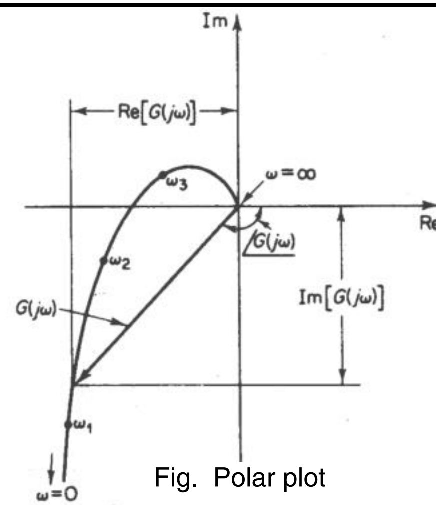


Fig. Polar plot

For two system connected in cascade, the overall transfer function is the product of combination. That is if $G(j\omega) = G_1(j\omega)\, G_2(j\omega)$ then the combination gives

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

where $|G(j\omega)| = |G_1(j\omega)|\,|G_2(j\omega)|$

and $\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$

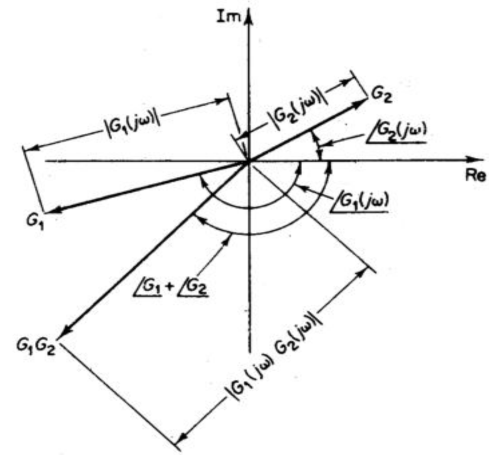
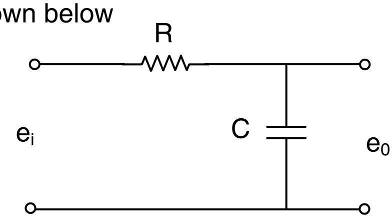


Fig. Polar plots of $G_1(j\omega)$, $G_2(j\omega)$, and $G_1(j\omega)\, G_2(j\omega)$

- For example :

  For the circuit shown below



$\therefore$ $$G(j\omega) = \frac{1}{1 + j\omega T}$$

$$= \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1} \omega T$$

To draw polar plot,

- $\omega = 0$, $M = 1$, $\phi = 0$
- ω increases, M decreases,

  ϕ increases negatively
- $\omega = \frac{1}{T}$, $M = \frac{1}{\sqrt{2}}$, $\phi = 45°$
- $\omega \to \infty$, $M \to 0$, $\phi = -90°$

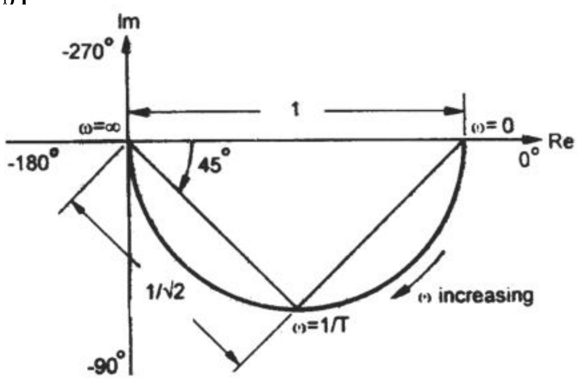
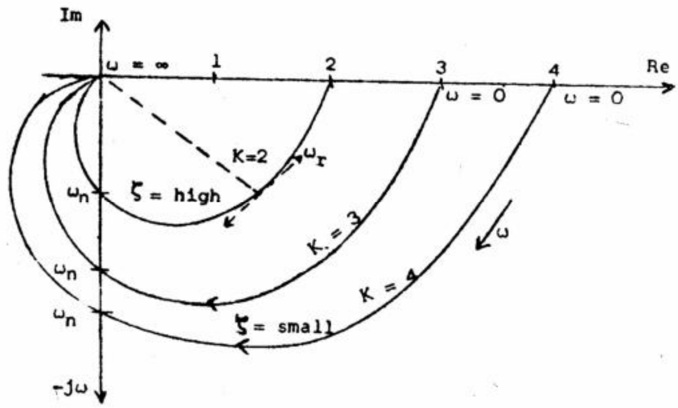


Fig. Polar plot of 1/(1 + jωT)

> The polar plot starts at a particular value when $\omega = 0$ and ends at 0 when $\omega \to \infty$.

The shape of polar plot depends on damping ration (ξ).



It reveals :

- The intersection point on negative imaginary axis is nothing but the undamped natural frequency ω_n.
- For high damping ratio, the plot is closer to the real axis.
- The plot tends to become semicircle for overdamped system.

Advantage

It incorporates entire frequency range in the same plot.

Disadvantage

It does not clearly indicates the contribution of individual factors.

- Integral and derivative factor :
  $\rightarrow$ The polar plot $G(j\omega) = 1/j\omega$ is the negative imaginary axis.

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ$$

  $\rightarrow$ The polar plot of $G(j\omega) = j\omega$ is the positive imaginary axis.

- First order factor :
  Consider first order sinusoidal transfer function

$$G(j\omega) = \frac{1}{1+j\omega T} \quad = \frac{1}{\sqrt{1+\omega^2 T^2}}\angle -\tan^{-1}\omega T$$

Polar plot :
$\omega = 0$, $M = 1$, $\phi = 0$
$\omega = \frac{1}{T}$, $M = \frac{1}{\sqrt{2}}$, $\phi = -45^\circ$
$\omega \to \infty$, $M \to 0$, $\phi \to -90^\circ$

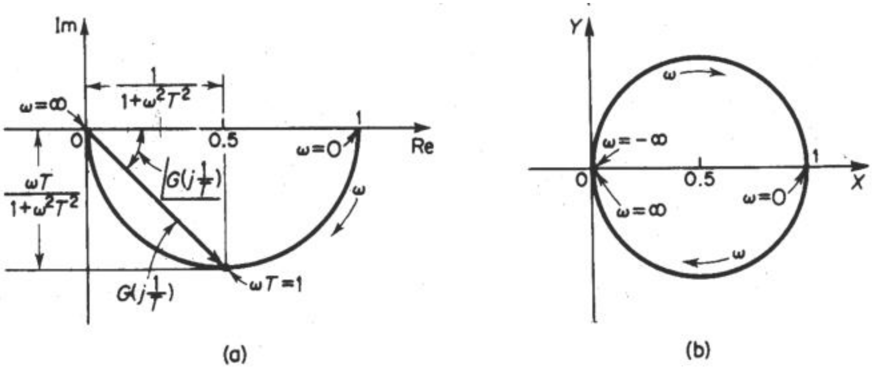


Fig. (a) Polar plot of 1/(1+jωT); (b) Plot of G(jω) in X–Y plane

Polar plot is a semicircle as frequency is varied from $\omega = 0$ to $\omega = \infty$

<u>Why the plot is semicircle</u> ?

Suppose $G(j\omega) = x + iy$

But $$G(j\omega) = \frac{1}{1+j\omega T}\left(\frac{1-j\omega T}{1-j\omega T}\right)$$

$$= \frac{1-j\omega T}{1+\omega^2T^2} = \frac{1}{1+\omega^2T^2} - j\frac{\omega T}{1+\omega^2T^2}$$

$$= x + iy$$

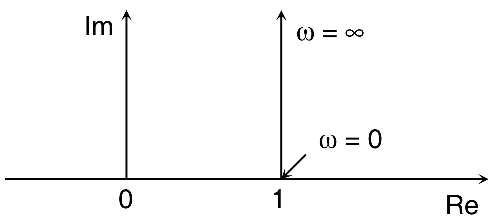
$\therefore$ $$\underset{\text{real part}}{x = \frac{1}{1+\omega^2T^2}}, \quad \underset{\text{imaginary part}}{y = \frac{-\omega T}{1+\omega^2T^2}}$$

taking centre point one the real axis at $\omega = 1/2$ then we obtain

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

This is the equation of circle with radius $\frac{1}{2}$ and center $\left(\frac{1}{2}, 0\right)$

The plot of transfer function 1 + jωT is simply upper half of the line passing through (1, 0)



Polar plot of the Quadratic factors :

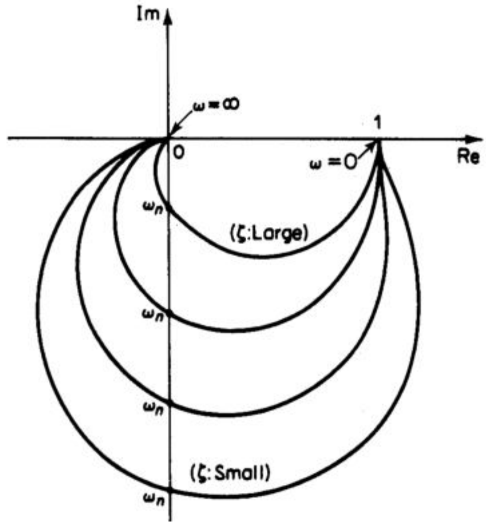


Fig. Polar plots of $1/\left(1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right)$

Plot starts at $1\angle 0°$ and ends at $0\angle -180°$.

For undamped case $\omega = \omega_n$, $G(j\omega_n) = \frac{1}{j2\xi}$ and $\phi = -90°$.

Peak value of $G(j\omega)$ is obtained as the ratio of the magnitude of vector at resonant frequency ω_r to magnitude of vector at $\omega = 0$.

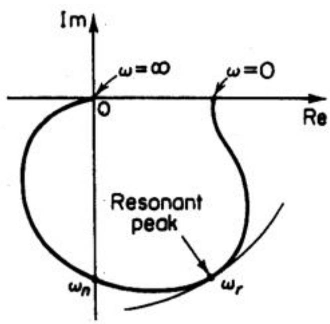
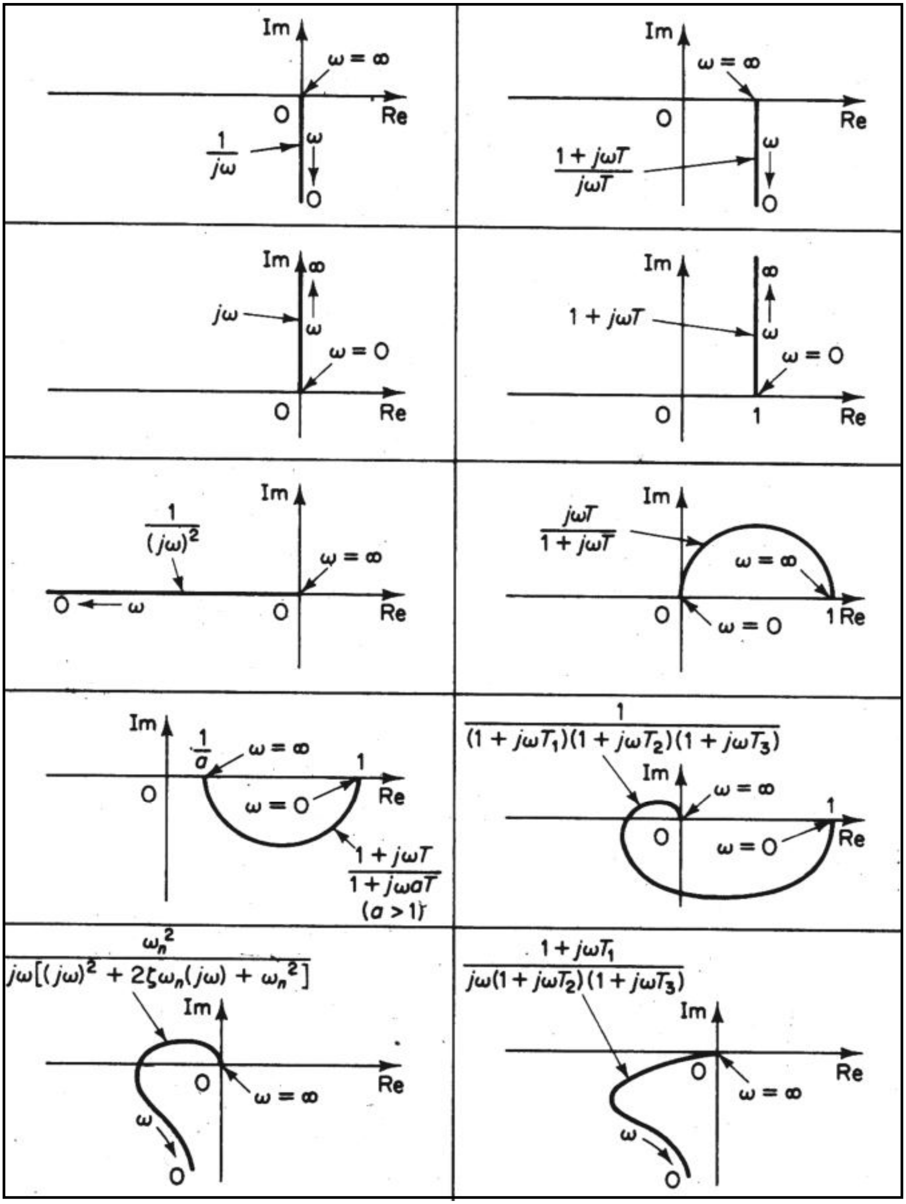


Fig. Polar–plot showing the resonant peak and resonant frequency ω_r

For overdamped case ξ increases beyond unity.

Table : Polar plots of Simple Transfer function



Inverse polar plot :

It is a graph of $\dfrac{1}{G(j\omega)}$ Vs ω. It is used in stability study of non unity feedback system.

Effect of adding s, s^2 type terms in the denominator :

Now consider $G_3(s) = \dfrac{1}{s(1+sT_1)}$.

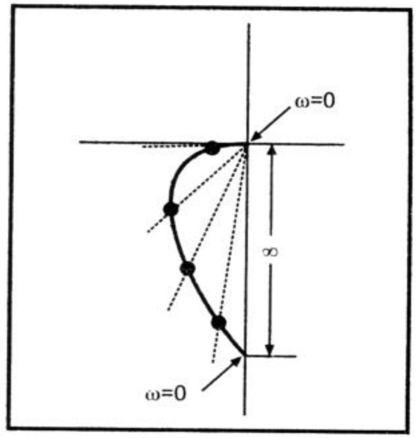


Fig. : Polar plot at $G_3(s)$

We shall compare the polar plot of $G_3(s)$ with that of

$$G(s) = \frac{1}{1+sT_1}$$

$$G_3(j\omega) = \frac{1}{j}\omega(1+j\omega T_1)$$

Here, $G_3|(j\omega)| = \dfrac{1}{\omega\times\sqrt{1+(\omega T_1)^2}}$

and $\phi = -\,90^\circ \tan^{-1}(\omega T_1)$

At $\omega = 0, \phi = -90^\circ$

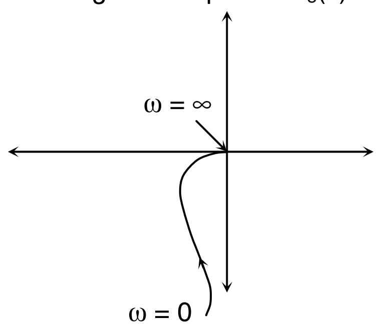
and at $\omega = \infty, \quad \phi = -180^\circ$

Rest of the facts are that at $\omega = 0$.

$|G(j\omega)| = \infty$

and at $\omega = \infty$, $|G(j\omega)| = 0$.

This gives the plot as shown. We have shown the general shape since the point at $\omega = 0$ cannot actually be drawn on a graph paper.



Now consider $G(s) = \dfrac{K}{s^2(1+sT_1)}$

$$G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)} \quad \text{giving} \quad |G(j\omega)| = \frac{1}{\omega^2\times\sqrt{1+(\omega T_1)^2}}$$

and $\phi = -2\times 90^\circ \tan^{-1}(\omega T_1) = -180^\circ - \tan^{-1}(\omega T_1)$ at $\omega = 0$, $|G(j\omega)| = 0$ and $\phi = -180^\circ$ and at $\omega = \infty$, $|G(j\omega)| = 0$ and $\phi = -270^\circ$

The magnitude of G $(j\omega)$ decreases and angle turns from -180° to -270° as ω increases giving the plot as shown.

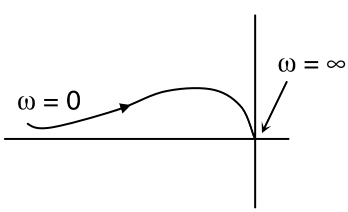
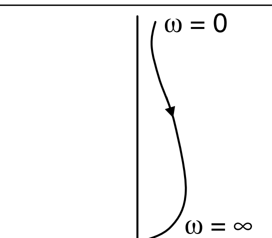


Fig : (a). Polar plot of $G(s) = \dfrac{K}{s^2(1+sT_1)}$



(b) Polar plot of $G_4(s)$

Similarly polar plot of $G_4(s) = \dfrac{K}{s^3(1+sT_1)}$ is also shown in figure (b).

General rule regarding poles

We can make two general rules :

1. The presence of each s term in denominator of G(s) shifts the staring point at $\omega = 0$ by $-90°$.
   Thus with no s term the plot starts at $\phi = 0°$ (on X–axis), with s term in denominator it starts at $\phi = -90°$ and with s^2 term at $\phi = -180°$ and so on.

2. Each simple pole term in the denominator, of the type $(1 + s\,T_1)$, adds $-90°$ rotation at $\omega = \infty$ (compared to that at $\omega = 0$).

Type – 1 : System with two and three poles :

The plots of $$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

and $$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)(1+sT_3)}$$ are shown

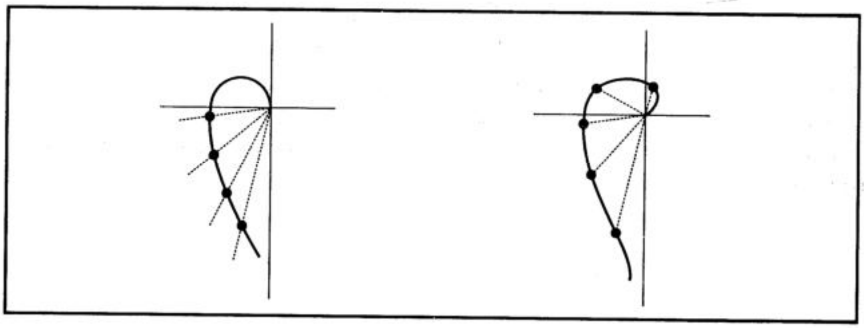


Fig. (a) Plot of $G_5(s)$ (b) : Plot of $G_6(s)$.

By adding pure zeros the starting point at $\omega = 0$ is changed by $+90°$.

For example the polar plots of $\frac{s}{1+sT_1}$ and $\frac{s^2}{1+sT_1}$ are as shown.

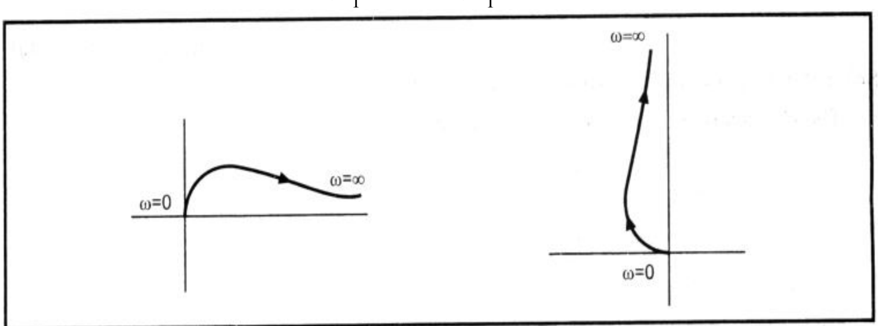


Fig.(a): Plot of $\frac{s}{1+sT_1}$ (b): Plot of $\frac{s^2}{1+sT_1}$

Effect of adding a term of the type (1 + s T) in the numerator :

Similarly if we add zero to the transfer function by multiplying it with a term of the type (1 + s T) in the numerator then the angle turned through is +90° when we change ω from zero to infinity.
For example plot of $G_7(s)$ and $G_8(s)$ are shown.

where $$G_7(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

and $$G_8(s) = \frac{K(1+sT_3)}{s(1+sT_1)(1+sT_2)}$$

The exact plots will depend on relative values of T_1, T_2 and T_3 and can only be determined by calculation.

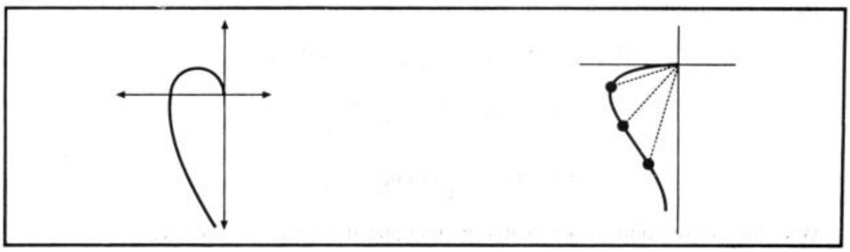
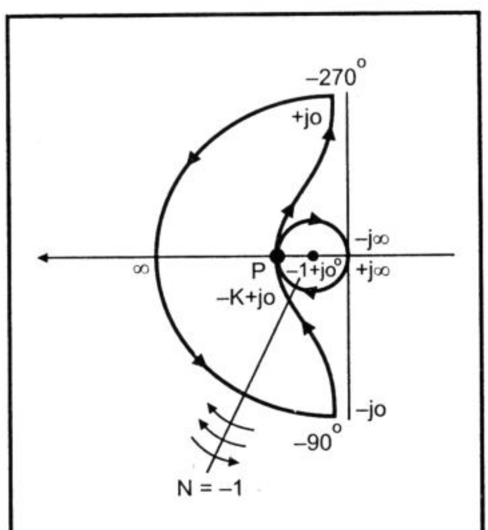


Fig.(a) Plot of $G_7(s)$ (b) Plot of $G_8(s)$

We find that at near about $\omega = \frac{1}{T_3}$ the magnitude tends to increase in magnitude and angle may change slowly with the change in ω. Adding a zero stabilities a system and makes it faster responding.

Intersection with negative real axis :



Rationalizing $G(j\omega)\,H(j\omega)$ we get,

$$G(j\omega)\,H(j\omega) = \frac{K(3+j\omega)(-j\omega)(-1-j\omega)}{(j\omega)(-j\omega)(-1+j\omega)(-1-j\omega)}$$

$$= \frac{-Kj\omega[-3-4j\omega+\omega^2]}{\omega^2(1+\omega^2)}$$

$$= \frac{-4\omega^2K}{\omega^2(1+\omega^2)} - \frac{Kj\omega[-3+\omega^2]}{\omega^2(1+\omega^2)}$$

For intersection with negative real axis,

$$-3+\omega^2 = 0$$

$$\therefore \quad \omega^2 = 3$$

$$\therefore \quad \omega_{pc} = 1.732 \text{ rad/sec.}$$

Hence intersection point say P is

$$P = \frac{-4K}{(1+3)} = -K$$

So Nyquist plot is as shown in figure.

For stability –1 + j 0 must be on right side of P.

$\therefore \quad |OP| > 1$

$\therefore \quad |-K| > 1$

$\therefore \quad K > 1$

So for all values of K > 1, system is stable.

Observation :

- addition of pole to a transfer function results in rotation of polar plot through angle –90° as $\omega \to \infty$
- addition of zero rotates polar plot high frequency portion by 90° in counter clockwise direction
- addition of pole at origin rotates polar plot at origin and ∞ by –90° further.

Nyquist Stability Criterion applied to Inverse Polar Plot :

For closed loop system to be stable, the encirclement of the plot of locus of $\frac{1}{G(s)H(s)}$ to the point –1 + 0 j must be counterclockwise. Also the number of encirclement should be equal to number of poles in right half. If open loop transfer function has no zeros in right half of s–plane then the encirclement number should be zero for closed loop system to be stable.

Note : If transfer function has exponential term i.e. transportation lag then the number of encirclement is infinite. So Nyquist stability criterion cannot be applied in this case.

The steady state response of a system to be sinusoidal input is called as Frequency response.

In LTI system, the frequency response is independent of amplitude and phase of input and also initial conditions.

Routh criterion is a time domain approach to check the stability condition. Root locus is a powerful time domain method for stability.

Nyquist criterion is powerful frequency domain method for stability check of a system.

For satisfactorily operation, the range of ξ is generally $0.4 < \xi < 0.707$

The nature of the roots of this equation determines absolute stability. If roots are on left half of s–plane then the system is stable.

If G(s) H(s) does not have any poles in the right half s–plane then Z = N. Thus there should be no encirclements of –1 + j0 point. The stability can be checked by seeing whether –1 + j0 points lie outside the shaded region or not. For stability –1 + j0 point must lie outside

Encirclement of −1 + j0 point is not sufficient to check whether multiple pole system is stable or not.

> For minimum phase system, both phase and gain margin should be positive. Negative margin indicates instability.

Observation :

- addition of pole to a transfer function results in rotation of polar plot through angle −90° as $\omega \to \infty$
- addition of zero rotates polar plot high frequency portion by 90° in counter clockwise direction
- addition of pole at origin rotates polar plot at origin and ∞ by −90° further.

> If transfer function has exponential term i.e. transportation lag then the number of encirclement is infinite.

So Nyquist stability criterion cannot be applied in this case.

BODE DIAGRAMS

A sinusoidal transfer function, a complex function of the frequency ω, is represented by two separate plots, one giving the magnitude versus frequency and the other the phase angle (in degrees) versus frequency. Both the plots are plotted against the frequency in logarithmic scale. The standard representation of the logarithmic magnitude of $G(j\omega)$ is 20 log $|G(j\omega)|$ where the base of logarithm is 10. The main advantage of using the logarithmic plot is that multiplication of magnitudes can be converted into addition.

The transfer function $G(j\omega)$ is represented by $e^{j\phi(\omega)}$
Taking natural log on both sides

$$\ln G(\omega) = \ln |G(j\omega)| + j\phi(\omega)$$

The real part is the natural logarithm of magnitude and is measured in a basic unit called neper; the imaginary part is the phase characteristic.
In Bode plot, the unit of magnitude 20 log $|G(j\omega)|$ is decibel (dB).

For an example, the transfer function, $G(s) = \dfrac{1}{1+sT}$

$$G(j\omega) = \frac{1}{\left(1+\omega^2T^2\right)^{1/2}} \angle -\tan^{-1}\omega T \qquad \text{....(1)}$$

The log–magnitude is

$$20\log|G(j\omega)| = 20\log\left(1+\omega^2T^2\right)^{-1/2}$$

$$= -10\log\left(1+\omega^2T^2\right) \qquad \text{....(2)}$$

For low frequencies $\left(\omega << \frac{1}{T}\right)$

$$20\ \log|G(j\omega)| = -10\log 1 = 0\,\text{dB} \qquad ....(3)$$

For high frequencies $\left(\omega >> \frac{1}{T}\right)$

$$20\ \log|G(j\omega)| = -20\log \omega T$$

$$= -20\log\omega - 20\log T \qquad ....(4)$$

A unit change in logω means

$$\log\left(\frac{\omega_2}{\omega_1}\right) = 1 \quad \text{or} \quad \omega_2 = 10\,\omega_1$$

This range of frequencies is called a decade.

The slope of the equation (4) is –20 dB/decade. The range of frequencies $\omega_2 = 2\ \omega_1$ is called an octave.

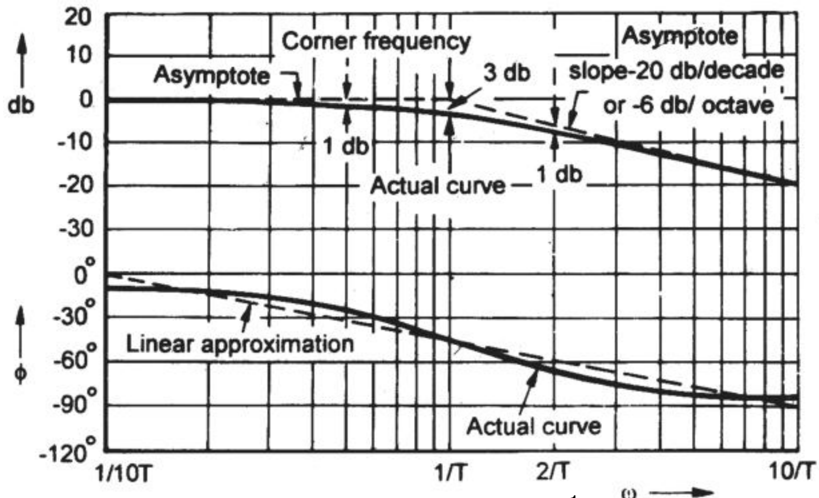


Fig.1 : Bode plot of $(1 + j\omega T)^{-1}$

The log magnitude versus log–frequency curve of (1/1+jωT) can be approximated by two straight line asymptotes, one straight line at 0 dB for frequency range $0 < \omega \leq 1/T$ and other, a straight line with a slope –20 dB/dec for the frequency range $1/T \leq \omega \leq \infty$. The frequency $\omega = \frac{1}{T}$ at which the two asymptotes meet is called the corner frequency or the break frequency.

- The corner frequency divides the plot in two regions, a low frequency region and a high frequency region.
- The log–magnitude plot of $(1 + j\omega T)^{-1}$ is an asymptotic approximation of the actual plot.

The error in log–magnitude for $0 < \omega \leq 1/T$ is given by

$$\boxed{-10 \log (1 + \omega^2 T^2) + 10 \log 1} \qquad ....(5)$$

Therefore, the error at the corner frequency $\omega = 1/T$ is

$$-10 \log(1 + 1) + 10 \log 1 = -3\text{dB} \qquad ....(6)$$

The error at frequency ($\omega = 1/2T$) one octave below the corner frequency is

$$-10 \log (1 + 1/4) + 10 \log 1 = -1 \text{ dB} \qquad ....(7)$$

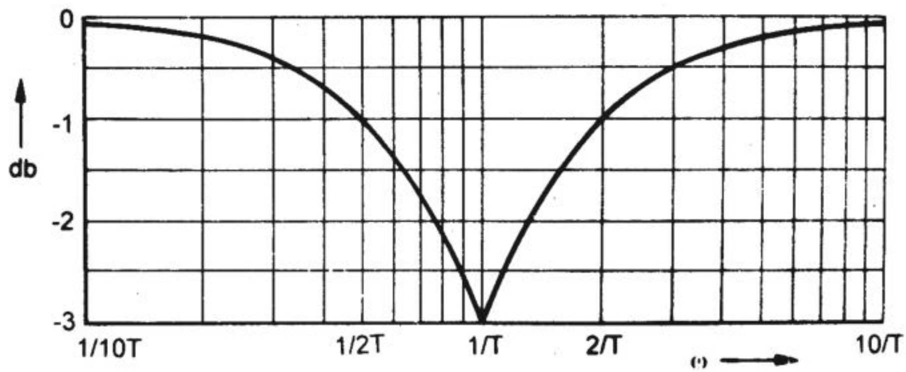


Fig. Error in log–magnitude versus frequency of $(1 + j\omega T)^{-1}$

The sinusoidal transfer function is

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

$$= \frac{1}{\sqrt{(1 + \omega^2 T^2)}} \angle -\tan^{-1} \omega T = M \angle \phi \qquad ....(8)$$

From equation (8), the phase angle ϕ of the factor $(1/(1 + j\omega T)$ is

$$\phi = -\tan^{-1} \omega T$$

At the corner frequency, the phase angle of this factor is

$$\phi = -\tan^{-1}\left(\frac{T}{T}\right) = -45^\circ$$

At zero frequency, $\phi = 0$ and at infinity, it becomes -90°.

Since the phase angle is given by inverse tangent function, the phase characteristic is skew symmetric about the inflection point $\phi = -45^\circ$.

Phase versus log–frequency plot can also be approximated by a straight line passing through -45° at the corner frequency ($\omega = 1/T$), 0° at $\omega = 1/10T$ and -90° at $\omega = 10/T$ as shown by dotted line in figure (1). Such an approximation has a maximum error of about 6°.

Consider the transfer function

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)}{(j\omega)^r(1+j\omega T_1)(1+j\omega T_2).....\left[1+2\xi\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]} \qquad ....(9)$$

The transfer function G(jω) has real zeros at $-1/T_a$, $-1/T_b$, a pole at the origin of multiplicity r, real poles at $-1/T_1$, $-1/T_2$, …. and complex poles at $-\xi\omega_n \pm j\omega_n\sqrt{\left(1-\xi^2\right)},...$

If the transfer function has complex zeros, quadratic terms of the form given in denominator will appear in the numerator as well.

The constant multiplier K is given by

$$K = \lim_{\omega\to 0}(j\omega)^r\, G(j\omega) \qquad ....(10)$$

where r = no. of poles of G(jω) at origin

= system type no.

Note : For type–0, type–1, type–2 systems,

$K = K_p$, K_v and K_a respectively.

From equation (10) the log–magnitude is given by

$$\begin{aligned} 20\log|G(j\omega)| = {} & 20\log K + 20\log|1+j\omega T_a| + 20\log|1+j\omega T_b| + \ldots\ldots \\ & - 20\, r\log(\omega) - 20\log|1+j\omega T_1| - 20\log|1+j\omega T_2| + \ldots. \\ & - 20\log|1+j2\xi\ (\omega/\omega_n) - (\omega/\omega_n)^2| \ldots\ldots \qquad ....(11) \end{aligned}$$

and the phase angle is given by

$$\begin{aligned} \angle G(j\omega) = {} & \tan^{-1}\omega T_a + \tan^{-1}\omega T_b + ... - r(90^\circ) - \tan^{-1}\omega T_1 \\ & - \tan^{-1}\omega T_2 - .... - \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^{\,2}}}\right\}...... \end{aligned}$$

The factors to be found in Bode Diagram are

1. Constant gain K
2. Poles at origin $(1/(j\omega)^r$
3. Pole on real axis $1/(1 + j\omega T)$
4. Zero on real axis $(1 + j\omega T)$
5. Complex conjugate poles $1/[1 + 2\xi\,(\omega/\omega_n)j - (\omega/\omega_n)^2\,]$
6. Complex conjugate zeros if present.

Factors of the form $K/(j\omega)^r$

The log magnitude of this factor is

$$20\log\left|\frac{K}{(j\omega)^r}\right| = -20r\log\omega + 20\log K$$

and the phase is

$$\phi(\omega) = -90^\circ\ r$$

General Procedure for Constructing Bode Plots

The following steps are generally involved in constructing the Bode plot for a given $G(j\omega)$.

1. Rewrite the sinusoidal transfer function in the time constant form as given in equation (9)
2. Identify the corner frequencies associated with each factor of the transfer function.
3. Knowing the corner frequencies, draw the asymptotic magnitude plot. This plot consists of straight line segments with line slope changing at each corner frequency by +20 db/decade for a zero and –20 db/decade for a pole (± 20 mdb/decade for a zero a pole multiplicity m). For a complex conjugate zero or pole the slope changes by ± 40 db/decade (± 40m db/decade for complex conjugate zero or pole of multiplicity m).
4. From the error graphs, determine the corrections to be applied to the asymptotic plot.
5. Draw a smooth curve through the corrected points such that it is asymptotic to the line segments. This gives the actual log–magnitude plot.
6. Draw phase angle curve for each factor and add them algebraically to get the phase plot.

Example

A unity feedback control system has $G(s) = \dfrac{80}{s(s+2)(s+20)}$

Draw the Bode Plot.

Solution

Step (1)

First arrange G(s) H(s) in time constant form.

$$G(s)\,H(s) = \frac{80}{s(s+2)(s+20)}$$

Here, H (s) = 1

$$G(s)\,H(s) = \frac{80}{2s\left(1+\frac{s}{2}\right)(20)\left(1+\frac{s}{20}\right)} = \frac{2}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{20}\right)}$$

Step (2)

Identify factors:

(i) k =2

(ii) 1 pole at origin

(iii) Simple pole $\dfrac{1}{\left(1+\dfrac{s}{2}\right)}$ with $T_1 = \dfrac{1}{2}$ $\quad \therefore \omega c_1 = \dfrac{1}{T_1} = 2$

(iv) Simple pole $\dfrac{1}{\left(1+\dfrac{s}{20}\right)}$ with $T_2 = \dfrac{1}{20}$ $\quad \therefore \omega c_2 = \dfrac{1}{T_2} = 20$

Step (3)

Magnitude Plot Analysis –

(i) For k =2, 20log k = 6dB.

(ii) For 1 pole at origin, straight line of slope –20dB/decade passing through intersection point of $\omega = 1$ & 0 dB.

(iii) For simple pole $\dfrac{1}{\left(1+\dfrac{s}{2}\right)}$ draw a line with slope –40db/decade at an intersection of $\omega_{c1} = 2$, this will continue till it intersects next corner frequency line i.e. $\omega_{c2} = 20$.

(iv) At $\omega_{c2} = 20$, there is another simple pole contributing –20dB/decade & hence the resultant slop after $\omega_{c2} = 20$ becomes –40 – 20 = –60 dB/decade.
This is resultant of overall G(s) H(s), i.e. G(jω) H(jω) and the final slope is –60dB/decade as no other factor is present.

Step (4)

Phase Plot

Convert G(s) H(s) to G (jω) H (jω)

$$\therefore G(j\omega)H(j\omega) = \frac{2}{j\omega\left(1+\dfrac{j\omega}{2}\right)\left(1+\dfrac{j\omega}{20}\right)}$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 2 + j0}{\angle j\omega \angle 1 + \dfrac{j\omega}{2} \angle 1 \angle \dfrac{j\omega}{20}}$$

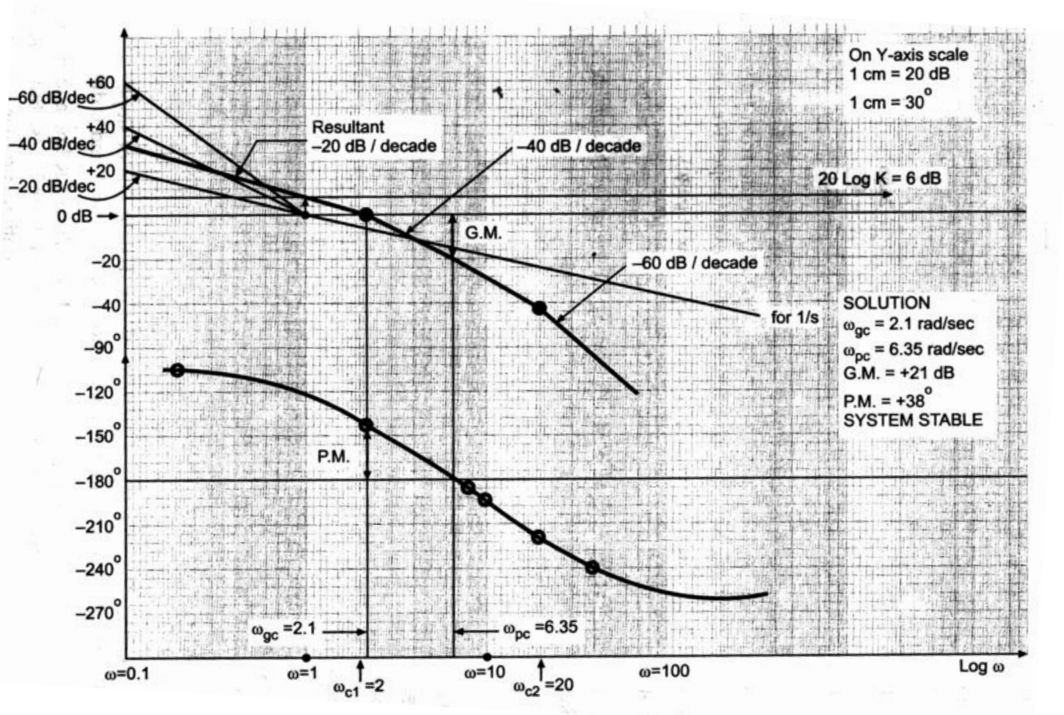
$\angle 2 + jo = 0^\circ$, $\quad \dfrac{1}{\angle j\omega}$ i.e 1 pole at origin is –90°.

$\angle \dfrac{1}{1+j\dfrac{\omega}{2}} = -\tan^{-1} + \dfrac{\omega}{2}$ and $\angle \dfrac{1}{1+j\dfrac{\omega}{20}} = -\tan^{-1}\dfrac{\omega}{20}$.

∴ Phase angle table is –

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\frac{\omega}{2}$	$-\tan^{-1}\frac{\omega}{20}$	ϕ_R Resultant
0.2	–90°	–5.7°	–0.57°	–96.27°
2	–90°	–4.5°	–5.7°	–140.7°
8	–90°	–75.96°	–21.8°	–187.7°
10	–90°	–78.69°	–26.56°	–195.29°
20	–90°	–84.28°	–45°	–219.28°
40	–90°	–87.13°	–63.43°	–240.58°
∞	–90°	–90°	–90°	–270°

From the Bode Plot, we found,
ω_{gc} = 2.1 rad/sec ω_{pc} = 6.35 rad/sec
G.M. = + 21dB P.M. = + 38°
For stable system, $\omega_{gc} < \omega_{pc}$ and G.M. and PM. are positive.
Thus the above system is stable.



System is said to be stable – when P.M. and G.M. are positive. i.e. $\omega_{gc} < \omega_{pc}$

System is said to be unstable – when both P.M. and G.M. are negative. i.e $\omega_{gc} = \omega_{pc}$

System is said to be marginally stable – when G.M. and P.M. both are zero. i.e. when $\omega_{gc} = \omega_{pc}$. This condition is useful to design marginally stable systems.

Complex Conjugate Poles

In normalized form, the quadratic factor for a pair of complex conjugate poles may be written as

$$\frac{1}{(1+j2\zeta u-u^2)}$$

where $u = \omega/\omega_n$ is the normalized frequency.

The log magnitude of this factor is

$$20\log\left|\frac{1}{1+j2\zeta u-u^2}\right| = -20\log\left[(1-u^2)^2+(2\zeta u)^2\right]^{1/2}$$

$$= -10\log\left[(1-u^2)^2+4\zeta^2u^2\right]$$

For $u << 1$, the log–magnitude is given by $20\log\left|\dfrac{1}{1+j2\zeta u-u^2}\right| \approx -10\log 1 = 0$

and for $u >> 1$, the log–magnitude is $20\log\left|\dfrac{1}{1+j2\zeta u-u^2}\right| \approx -20\log u^2 = -40\log u$

Therefore, the log–magnitude curve of the quadratic factor under consideration, consists of two straight line asymptotes, one horizontal line at 0 dB for $u << 1$ and the other, a line with a slope –40 dB/dec for $u >> 1$. These two asymptotes meet at 0 dB line i.e. $\omega = \omega_n$, corner frequency.

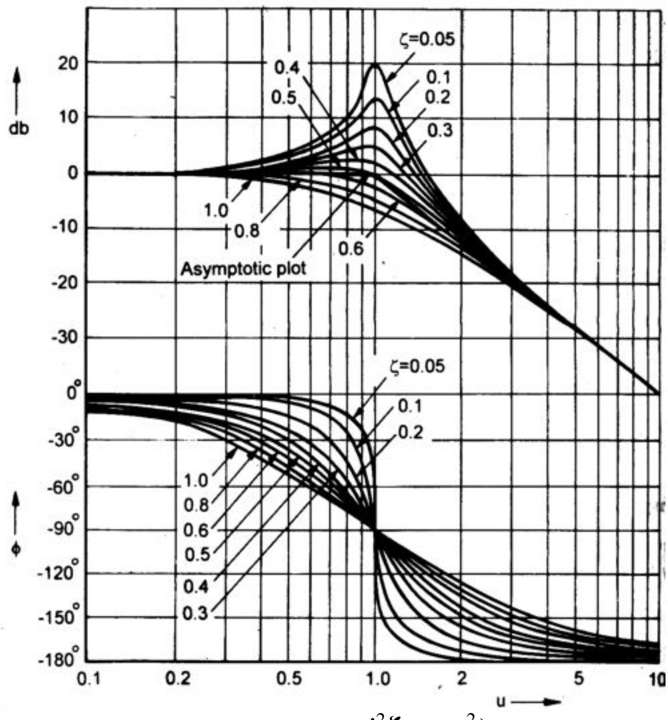


Fig. Bode plot of $1/(1 + j^2\zeta u - u^2)$

The error between the actual magnitude and the asymptotic approximation is given below
For 0 < u <<1, the error is

$$-10 \log [(1-u^2)^2 + 4\xi^2 u^2] + 10 \log 1$$

and for 1 < u << ∞, the error is

$$-10 \log [(1-u^2)^2 + 4\xi^2 u^2] + 40 \log u$$

The phase angle of the quadratic factor $1/(1 + j2\xi u - u^2)$ is given by

$$\phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

The phase angle plots for various values of ζ are given in figure above. All these plots have a phase angle of 0° at u = 0, –90° at u = 1 and –180° at u = ∞. The curves become sharper in going from low frequency range to the high frequency range as ζ decreases until for $\zeta = 0$ the curve jumps discontinuously from 0° down to –180° at u = 1.

In the above discussion, the plots of factors $(j\omega)^r$, i.e., zeros at the origin and $[1 + j2\zeta(\omega/\omega_n) - (\omega/\omega_n)^2]$, i.e., complex zeros, are not considered. The plots of these factors are similar to the plots of poles at the origin $1/(j\omega)^r$ and complex poles $1/[1 + j\zeta(\omega/\omega_n) - (\omega/\omega_n^2)]$, respectively, but with opposite signs.

MINIMUM PHASE SYSTEMS AND NON–MINIMUM PHASE SYSTEMS

A system having all the poles and zeros in the left half s–plane is called minimum–phase system, whereas those having poles and/or zeros in the right–half s–plane are non–minimum phase system.

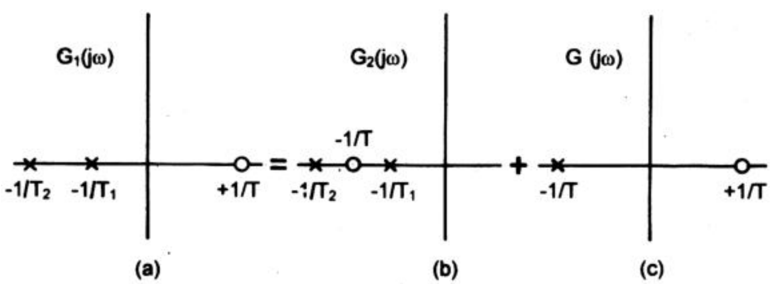


Fig. Pole–zero patterns for (a) non-minimum phase function (b) minimum–phase function; (c) all–pass function

- For systems with the same magnitude characteristic, the range of phase angle of minimum phase transfer function is minimum while the range in phase angle of any non–minimum phase transfer function is greater than this minimum.
- For a minimum phase system, the magnitude and phase angle characteristics are uniquely related. It means that if the magnitude plot is given for the frequency range 0 to infinity, then the phase plot is uniquely determined and vice versa.
  This does not hold for non-minimum phase system.

Identifying minimum and non–minimum phase system from Magnitude and Phase plot

- For a minimum phase system, the phase angle at $\omega = \infty$ becomes $-90°\ (q - p)$, where p and q are degrees of the numerator and denominator polynomials of the transfer function, respectively. For non-minimum phase system, the phase angle of $\omega = \infty$ differs from $-90°\ (q - p)$.

- In either system, the slope of the log–magnitude curve at $\omega = \infty$ is equal to $-20\ (q - p)$ dB/dec.

Transport Lag

Transport lag is of non-minimum phase behavior has an excessive phase lag with no attenuation at high frequencies. Such transport lag normally exists in thermal, hydraulic and pneumatic systems.

$$G(j\omega) = e^{-j\omega T}$$

The magnitude is always equal to unity since

$$|\, G(j\omega)\, | = |\, \cos \omega T - j \sin \omega T\, | = 1$$

Therefore the log magnitude of the transport lag $e^{-j\omega T}$ is equal to 0 dB. The phase angle of transport lag is

$$\angle G(j\omega) = -\omega T\ \ (\text{rad})\ = -57.3\ \ \omega T\ (\text{degrees})$$

The phase angle varies linearly with ω.

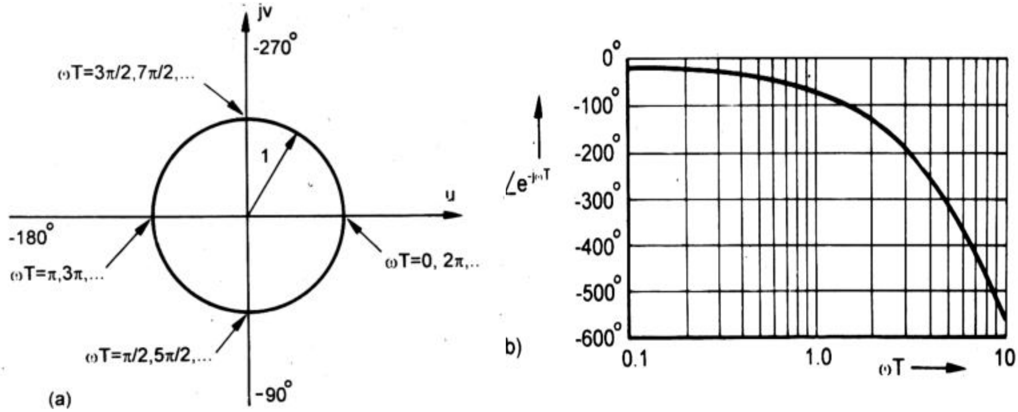


Fig. Phase angle characteristics of $e^{-j\omega T}$

The type of the system determines the slope of the log–magnitude curve at low frequencies.

- **Determination of Static Position Error Constants**

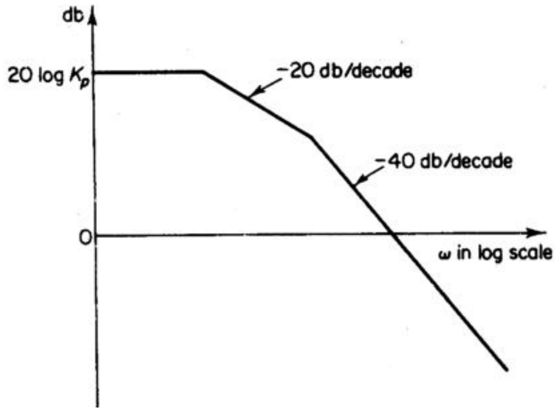


Fig. Log–magnitude curve of a type 0 system

In above system, the magnitude of G(jω) H(jω) equals K_p at low frequencies, or

$$\lim_{\omega \to 0} G(j\omega)\,H(j\omega) = K_p$$

- **Determination of Static Velocity Error Constants**

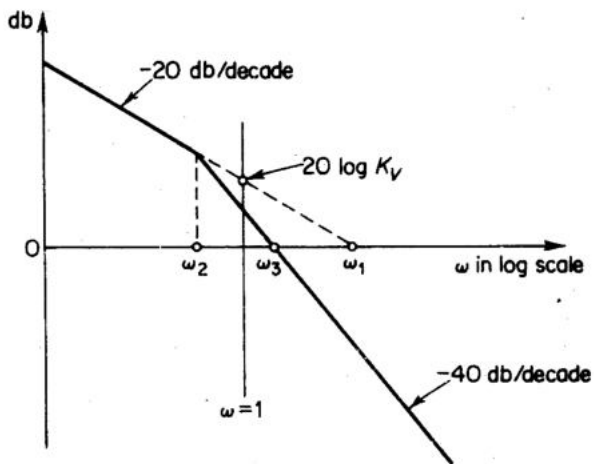


Fig. Log–magnitude curve of a type 1 system

From the above diagram, it is seen that the intersection of the initial –20 dB/dec segment with the line ω = 1 has the magnitude 20 log K_v.

In type 1 system

$$G(j\omega)\,H(j\omega) = \frac{K_v}{j\omega} \quad \text{for } \omega << 1$$

Thus, $$20\log\left|\frac{K_v}{j\omega}\right|_{\omega=1} = 20\log K_v$$

The intersection of the initial – 20 dB/dec segment with 0 dB line has frequency equal to K_v.

$$\left|\frac{K_v}{j\omega_1}\right| = 1 \qquad \text{or} \qquad K_v = \omega_1$$

- **Determination of Static Acceleration Error Constants**

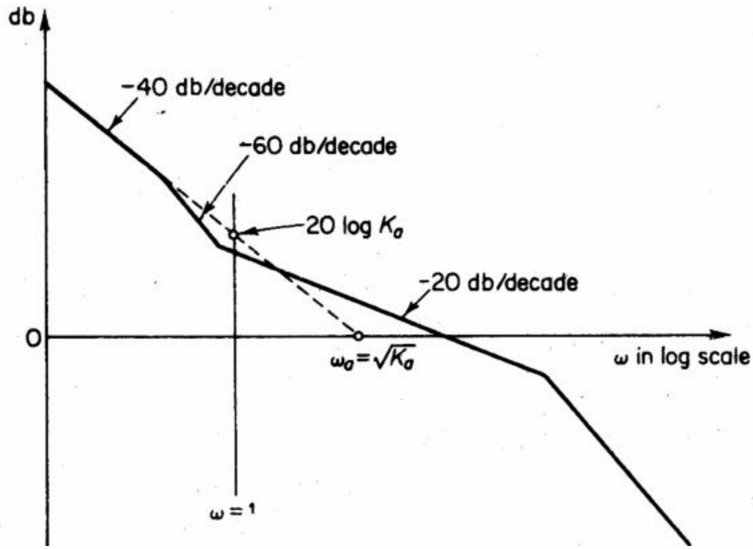


Fig. Log magnitude curve of a type 2 system.

The intersection of the initial –40 dB/dec segment with the $\omega = 1$ line has the magnitude of 20 log K_a

At low frequencies

$$G(j\omega)\,H(j\omega) = \frac{K_a}{(j\omega)^2}$$

it follows that

$$20\log\left|\frac{K_a}{(j\omega)^2}\right|_{\omega=1} = 20\log K_a$$

The frequency ω_a at the intersection of the initial –40 dB/dec. Segment with the 0 –dB line gives the square root of K_a numerically.

$$20\log\left|\frac{K_a}{(j\omega_a)^2}\right| = 20\log 1 = 0$$

which yields $\omega_a = \sqrt{K_a}$

LIST OF FORMULAE

- As $\xi \to 0$, $\omega_r \to \omega_n$, $M_r \to \infty$
- As $0 < \xi < 0.707$, $\omega_r < \omega_n$ always, $M_r > 1$
- As $\xi > 0.707$, magnitude of M decreases monotonically from 1 to 0.
- Denormalized bandwidth $\omega_b = \omega_n \left[1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}\right]^{1/2}$
- The relation between phase margin and damping ratio $\xi = \dfrac{\gamma}{100}$

LMR (LAST MINUTE REVISION)

- The main advantage of using the logarithmic plot is that multiplication of magnitudes can be converted into addition.

- The real part is the natural logarithm of magnitude and is measured in a basic unit called neper; the imaginary part is the phase characteristic.

- The corner frequency divides the plot in two regions, a low frequency region and a high frequency region.

- The log–magnitude plot of $(1 + j\omega T)^{-1}$ is an asymptotic approximation of the actual plot.

- – System is said to be stable – when P.M. and G.M. are positive. i.e. $\omega_{gc} < \omega_{pc}$
  – System is said to be unstable – when both P.M. and G.M. are negative. i.e $\omega_{gc} = \omega_{pc}$
  – System is said to be marginally stable – when G.M. and P.M. both are zero. i.e. when $\omega_{gc} = \omega_{pc}$. This condition is useful to design marginally stable systems.

- A system having all the poles and zeros in the left half s–plane is called minimum–phase system, whereas those having poles and/or zeros in the right–half s–plane are non–minimum phase system.

- For systems with the same magnitude characteristic, the range of phase angle of minimum phase transfer function is minimum while the range in phase angle of any non–minimum phase transfer function is greater than this minimum

- For a minimum phase system, the magnitude and phase angle characteristics are uniquely related. It means that if the magnitude plot is given for the frequency range 0 to infinity, then the phase plot is uniquely determined and vice versa. This does not hold for non minimum phase system.

- The type of the system determines the slope of the log–magnitude curve at low frequencies.

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COMPENSATION TECHNIQUE

A device inserted into the system for the purpose of satisfying the specifications is called a compensator.

There are three types of compensators used

- lead compensator
- lag compensator
- lag–lead compensator

If a sinusoidal input ℓ_i is applied to the input of a network and the steady–state output ℓ_o has a phase lead, then the network is called lead network.

If the steady state output e_o has a phase lag, then the network is called a lag network.

In lag–lead network, both phase lag and phase lead occur in the output but in different frequency regions.

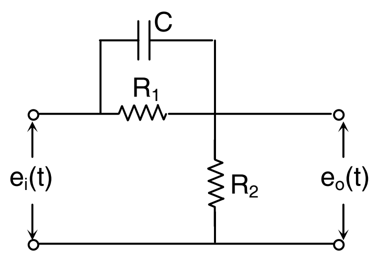
> In lag–lead network, phase lag occurs in the low frequency region and phase lead occurs in the high frequency region.

→ Information obtainable from open loop frequency response –
The low frequency region of the locus indicates the steady state behaviour of the closed loop system.

The medium frequency region of the locus indicates relative stability. The high frequency region indicates the complexity of the system.

Lead Compensator

Let us consider the following circuit :



In Laplace domain, the transfer function is

$$\frac{E_o(s)}{E_i(s)} = \frac{\alpha(1+Ts)}{(1+\alpha Ts)}$$

where,
Time constant T = RC

$$\alpha = \frac{R_2}{R_1+R_2} < 1$$

Maximum Lead Angle (ϕ_m)

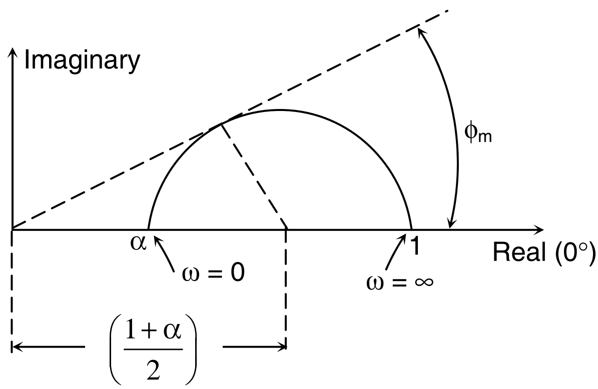
$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\phi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

ϕ_m is also given as

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right)$$

Phasor Plot of Lead Compensator



Lead compensation has the following effects

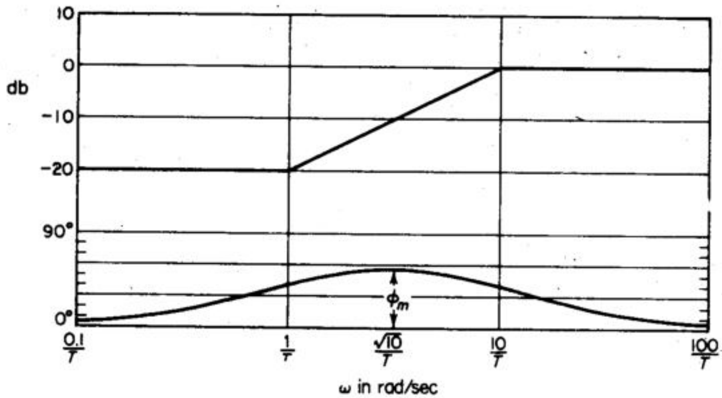
a) Bandwidth of closed loop system increases.
b) It increases damping by adding a dominant zero and a far off pole.
c) Phase margin improves, while steady state error does not get affected.

Characteristics

1. Improves transient response and a small change in steady state accuracy.
2. It may increase high frequency noise effect.

From the above transfer function, it is clear that lead compensator has a zero at s = $\frac{-1}{T}$ and a pole at S = $-\frac{1}{\alpha T}$. Since $0 < \alpha < 1$, zero is always located to the right of the pole in the complex plane. The minimum value of α is limited by the physical construction of the lead compensator. The minimum value of α is usually taken b to be about 0.07.

> The maximum phase lead that may be produced by a lead compensator is about 60°.



The above diagram shows the Bode–diagram of a lead compensator with $k_c = 1$ & $\alpha = 0.1$.

The corner frequencies for the lead compensator are w = 1/T and w = $\dfrac{1}{(\alpha T)} = \dfrac{10}{T}$

From the figure, W_m is the geometric mean of the two corner frequencies or

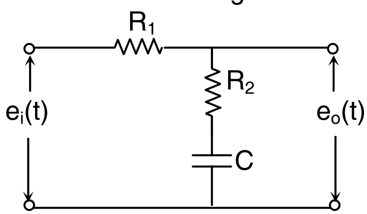
$$\log w_m = \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha T}\right)$$

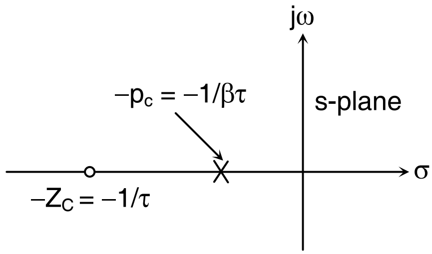
Hence $$w_m = \frac{1}{\sqrt{\alpha}T}$$

> Lead compensator is a high pass filter.

Lag Compensator

Let us consider the following network :





In Laplace domain, the transfer function is

$$\frac{E_o(s)}{E_i(s)} = \frac{1+Ts}{1+\beta Ts}$$

where,

$$T = RC$$

$$\beta = \frac{R_1 + R_2}{R_2} > 1$$

> The phase lag angle does not play a role in lag compensation.

The lag compensation has following effects :

a) Bandwidth of the system decreases which leads to an increase in rise time and settling time.
b) It uses attenuator characteristics for compensation.
c) It behaves as a PI controller and hence stability of the system decreases.

Characteristics

1. Appreciably improves steady state accuracy but increases transient response time.
2. It suppresses high frequency noise effects.

In the s–plane of lag compensator, it is shown that the pole is at $-\frac{1}{\beta\tau}$ and a zero is at $-\frac{1}{\tau}$ with the zero located to the left of the pole on the negative real axis.

The general form of transfer function of Lag compensator is

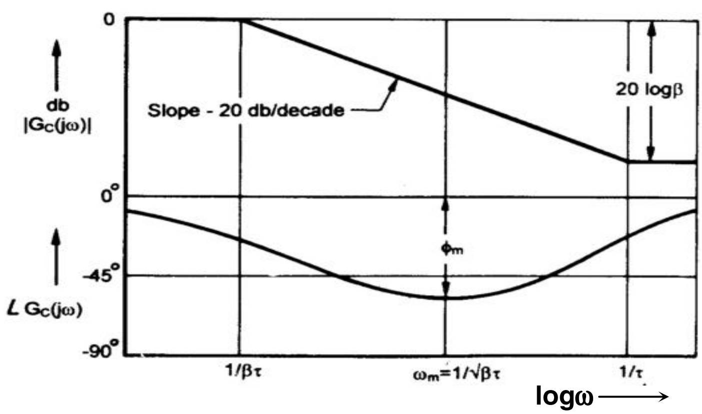
$$G_C(s) = \frac{s+z_c}{s+p_c} = \frac{s+\frac{1}{\tau}}{1+\frac{1}{\beta\tau}} = \frac{1+\tau s}{1+\beta\tau s} \; ; \qquad \beta\frac{z_c}{p_c} > 1, \tau > 0$$

The sinusoidal transfer function of the lag n/w is given by

$$G_c(j\omega) = \frac{1+j\omega\tau}{1+j\beta\omega\tau}$$

Since $\beta > 1$, the steady state output has a lagging phase angle with respect to the sinusoidal input.

The bode diagram is shown below :



The maximum phase lag ϕ_m is given as

$$\phi_m = \sin^{-1}\frac{(1-\beta)}{(1+\beta)}$$

where $\beta = \frac{R_1+R_2}{R_2} > 1$

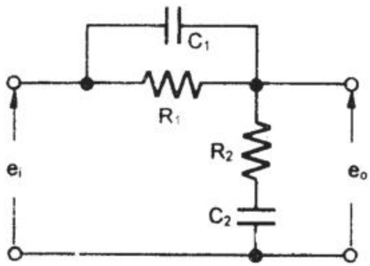
The maximum frequency ω_m is given as

$$\omega_m = \frac{1}{\left(\tau\sqrt{\beta}\right)} = \sqrt{\left(\frac{1}{\tau}\right)\left(\frac{1}{\beta\tau}\right)}$$

From the bode plot, it is seen that :

- The lag network has a dc gain of unity while it offers a high frequency gain of $1/\beta$.
- Since $\beta > 1$, it means that the high frequency noise is attenuated whereby the signal to noise ratio is improved.
- Typical choice of β is 10.

Lag Lead compensator



It is a combination of a lag & lead compensator.

$$G_c(s) = \underbrace{\left(\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\beta\tau_1}}\right)}_{\text{Lag section}}\underbrace{\left(\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\alpha\tau_2}}\right)}_{\text{Lead section}} \quad ;\ \beta > 1,\quad \alpha < 1 \qquad \ldots(1)$$

$$\frac{E_o(s)}{E_i(s)} = \left[\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right)s+\frac{1}{R_1R_2C_1C_2}}\right] \qquad \ldots(2)$$

From above equations (1) & (2) we have

$$R_1C_1 = \tau \qquad \ldots(3)$$

$$R_2C_2 = \tau_2 \qquad \ldots(4)$$

$$R_1C_1R_2C_2 = \alpha\beta\tau_1\tau_2 \qquad \ldots(5)$$

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2} = \frac{1}{\beta\tau_1}+\frac{1}{\alpha\tau_2} \qquad \ldots(6)$$

From equations (3), (4), (5), (6) it is found that

$$\alpha\beta = 1$$

$$G_c(s) = \frac{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau_2}\right)}{\left(s+\frac{1}{\beta\tau_1}\right)\left(s+\frac{\beta}{\tau_2}\right)}; \qquad \beta > 1$$

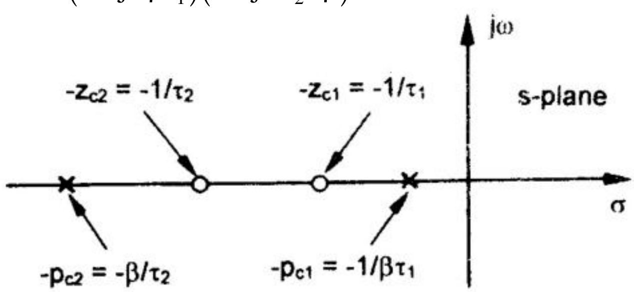
$$= \left(\frac{s+zc_1}{s+pc_1}\right)\left(\frac{s+zc_2}{s+pc_2}\right) \quad ; \qquad \beta = \frac{zc_1}{pc_1} = \frac{pc_2}{zc_2} > 1$$

where $\tau_1 = R_1C_1$, $\tau_2 = R_2C_2$, and $\beta > 1$ such that

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}=\frac{1}{\beta\tau_1}+\frac{\beta}{\tau_2}$$

The sinusoidal transfer function of lag–lead compensator is given by

$$G_c(j\omega) = \frac{(1+j\omega\tau_1)(1+j\omega\tau_2)}{(1+j\omega\beta\tau_1)(1+j\omega\tau_2/\beta)}$$



General Effects of Lead Compensator

1. Lead compensator raises the order of the system by one.
2. Zero is located to the right of pole and nearer to the origin.
3. Lead compensator reduces damping i.e. overshoot, rise time and settling time.
   Overall it increases speed of the response.
4. It improves phase and gain margins improving the relative stability.
5. It increases bandwidth of the system.

Limitations

1. If more than 60° phase lead is required then, multistage lead compensators need to be used.
2. To attenuate the offset, larger gain is required which in turn increases the required space, weight & cost of the system.
3. Large bandwidth may make the system more susceptible to noise because of high frequency gain.

General Effects of Lag Compensator

1. Lag compensator is basically a low pass filter. Thus it allows high gain at low frequencies and improves the steady state response.
2. In lag compensation, the attenuation characteristics are used for the compensation.
3. Lag compensator reduces gain cross over frequency ω_{gc}, and so bandwidth.
4. Reduction in bandwidth causes increases in rise time and settling time.
5. System becomes more sensitive to parameter variations and less stable.

General Effects of Lag Lead Compensator

1. It improves bandwidth or response time as well as low frequency gain which improves the steady state.
2. It uses the characteristics of both, lead compensation and lag compensation.

> The use of lead or lag compensators raises the order of the system by one.
> The use of lag–lead compensator raises the order of the system by two.

CONTROLLERS

There are 4 types of controllers.

1. **PD Type Controller**

   This type of controller changes the controller input to proportional plus derivative of error signal. It is used in the forward path. Proportional plus derivative controller has following effects on the systems :

   a) The peak overshoot and settling time reduces.

   b) There is no change in steady state error.

   $$m(t) = e(t) + \frac{de(t)}{dt}$$

   c) 'TYPE' and 'ω_n' remains unchanged

   d) The damping ratio increases.

> There is an improvement in transient part when PD controller is used.

2. **PI Type Controller**

   This type of controller changes the controller input to proportional plus integral of error signal. It is used in the forward path. Proportional plus integral controller increases 'TYPE' and 'ORDER' of the system.

   $$m(t) = e(t) + \int e(t)\,dt$$

> Steady state error reduces i.e. steady state part improves when PI controller is used.

3. **PID Type Controller**

As PD improves transient part and PI improves steady state part; thus, overall time response of the system improves drastically.

$$m(t) = e(t) + \frac{de(t)}{dt} + \int e(t)\,dt$$

4. **Rate Feedback Controller**

Rate feedback controller is known as output derivative controller. It is also called minor feedback loop compensation. In this, the derivative of output signal is feedback and compared with signal proportional to error.

LIST OF FORMULAE

- Maximum Lead Angle (ϕ_m)

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\phi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

ϕ_m is also given as

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right)$$

- The maximum phase lag ϕ_m is given as

$$\phi_m = \sin^{-1}\frac{(1-\beta)}{(1+\beta)}$$

where $\beta = \frac{R_1 + R_2}{R_2} > 1$

- The maximum frequency ω_m is given as

$$\omega_m = \frac{1}{\left(\tau\sqrt{\beta}\right)} = \sqrt{\left(\frac{1}{\tau}\right)\left(\frac{1}{\beta\tau}\right)}$$

LMR (LAST MINUTE REVISION)

- A device inserted into the system for the purpose of satisfying the specifications is called a compensator.
- If a sinusoidal input ℓ_i is applied to the input of a network and the steady–state output ℓ_o has a phase lead, then the network is called lead network.
- If the steady state output e_o has a phase lag, then the network is called a lag network.
- The medium frequency region of the locus indicates relative stability. The high frequency region indicates the complexity of the system.
- Lead network has increased bandwidth, increased damping ratio and improved phase Margin.
- The minimum value of α is limited by the physical construction of the lead compensator. The minimum value of α is usually taken to be about 0.07.
- The phase lag angle does not play a role in lag compensation. Attenuation at high frequency is used for compensation.
- The lag compensation decreases the bandwidth of the system, and increases rise time and also decreases relative stability.
- The lag network has a dc gain of unity while it offers a high frequency gain of $1/\beta$.
- Lead compensator raises the order of the system by one.
- Zero is located to the right of pole and nearer to the origin.
- Lag compensator is basically a low pass filter. Thus it allows high gain at low frequencies and improves the steady state response.
- The use of lead or lag compensators raises the order of the system by one. The use of lag–lead compensator raises the order of the system by two.
- Proportional plus derivative controller reduces the peak overshoot and settling time.
- As PD improves transient part and PI improves steady state part; thus, overall time response of the system improves drastically in case of *PID Type Controller.*

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Topic 7 : Mathematical Modelling and State Space Analysis

INTRODUCTION TO MATHEMATICAL MODELLING

A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately.

A mathematical model is not unique for a given system.

It is possible to improve the accuracy of a mathematical model by increasing its complexity.

> Linear system : A system is called linear if the principle of superposition applies.

Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results.

MECHANICAL SYSTEMS

Equivalent Mechanical Systems

a) The terms for an element connected to a node 'X' and stationary surface (reference) is

For mass $\rightarrow$ $M\dfrac{d^2x}{dt^2}$

For friction $\rightarrow$ $B\dfrac{dx}{dt}$

For spring $\rightarrow$ kx

b) The term for an element connected between the two nodes 'x_1' and 'x_2' i.e. between surfaces is

For friction $\rightarrow$ $B\left[\dfrac{dx_1}{dt}-\dfrac{dx_2}{dt}\right]$

For spring $\rightarrow$ $k\left[x_1-x_2\right]$

No mass can be between the two nodes as due to mass there cannot store potential energy

c) All elements which are under the influence of same displacement get connected in parallel under than node indicating the corresponding placement.

Gear Trains

The gear train is a device that transmits energy from one part of a system to another in such a way that force, speed and displacement may be altered. The inertia and friction of the gears are neglected in the ideal case. Consider a gear system as shown below.

The number teeth on the surface of the gears is proportional to the radii r_1 and r_2 of the gears

i.e. $r_1N_2 = r_2N_1$

💡	The distance traveled along the surface of each gear is same i.e. $\theta_1 r_1 = \theta_2 r_2$

The work done by one gear is same as the other i.e. $T_1\theta_1 = T_2\theta_2$

$\therefore$ we can say $$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Points to be noted :

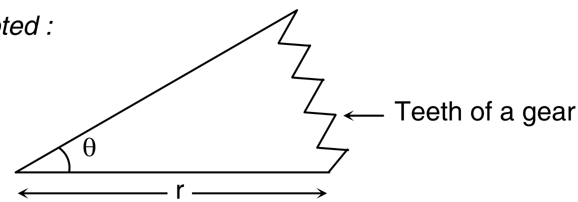
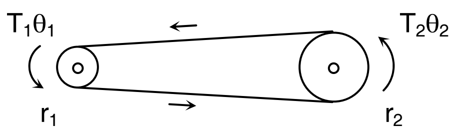


Fig. Part of a gear

1) The number of teeth N are proportional to the radius r of a gear.
2) The distance traveled on each gear is same
3) Work done = $T\theta$ by each gear is same.

Belt or Chain Drives

Belt and chain drives does the same function as that of gear train.



Assuming that there is no slippage between belt and pulleys we can write,

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} \quad \text{for such drive}$$

- **Mechanical Translation Systems**

Consider a spring–mass–dashpot system. Dash pot provides viscous friction or damping. The dashpot absorbs energy and dissipates it as heat. The dashpot is also called as a damper.

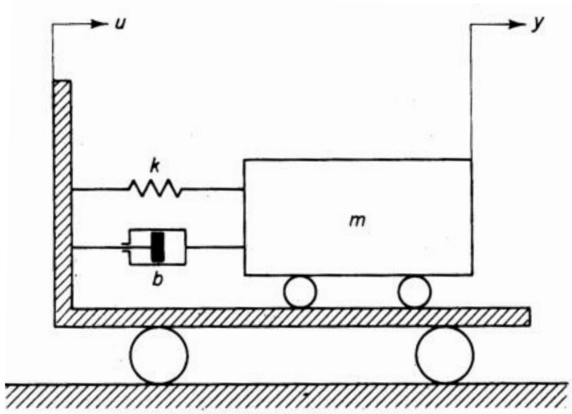


Fig. Spring–mass–dashpot system mounted on a cart

In this system, u(t) is the displacement of the cart.

At t = 0, the cart is moved at a constant speed, or $\dot{u}$ = constant. The displacement y(t) of the mass is the output.

In this system, m = mass, B = viscous friction, k = spring constant.

We assume that the friction force of damper is proportional to $\dot{y}-\dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to y – u.

For Translational system, Newton's second law states that

$$ma = \sum F$$

where m = mass, kg
a = acceleration, m/sec^2
F = force, N

Applying Newton's second law to the present system, we obtain

$$m\frac{d^2y}{dt^2} = -B\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

or
$$m\frac{d^2y}{dt} + B\frac{dy}{dt} + ky = B\frac{du}{dt} + ku \qquad \text{....(A)}$$

Above equation gives a mathematical model of the system considered.

A transfer function model is another way of representing a mathematical model of a liner, time–invariant system. For the present mechanical system, the transfer function model can be obtained as follows : Taking the Laplace transform of each term of Equation (A) gives

$$\lambda\left[m\frac{d^2y}{dt^2}\right]=m\left[s^2Y(s)-sy(0)-\dot{y}(0)\right]$$

$$\lambda\left[b\frac{dy}{dt}\right]=b\left[sY(s)-y(0)\right]$$

$$\lambda[ky]\;=kY(s)$$

$$\lambda\left[b\frac{du}{dt}\right]=b\left[sU(s)-u(0)\right]$$

$$\lambda[ku]=kU(s)$$

If we set the initial conditions equal to zero, or set y(0) = 0, $\dot{y}(0)=0,$ and $u(0)=0,$ the Laplace transform of Equation (A) can be written as

$$\left(ms^2+Bs+k\right)Y(s)=(Bs+k)U(s)$$

Taking the ratio of Y(s) to U(s), we find the transfer function of the system to be

$$\text{Transfer function}=G(s)=\frac{Y(s)}{U(s)}=\frac{Bs+k}{ms^2+Bs+k}$$

- **Mechanical Rotational System**

Consider the system shown below

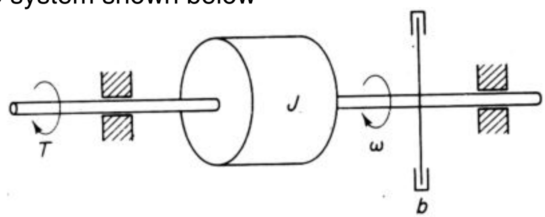


Fig. Mechanical rotational system

The system consists of a load inertia and a viscous friction damper.
For such a mechanical rotational system, Newton's second law states that

$$J\alpha=\sum T$$

where J = moment of inertia of the load, kg–m^2
α = angular acceleration of the load, rad/sec^2
T = torque applied to the system, N–m

Applying Newton's second law to the present system, we obtain

$$J\dot{\omega}=-B\omega+T$$

where J = moment of inertia of the load, kg–m^2
b = viscous–friction coefficient, N–m/rad/sec
ω = angular velocity, rad/sec
T = torque, N–m

The last equation may be written as

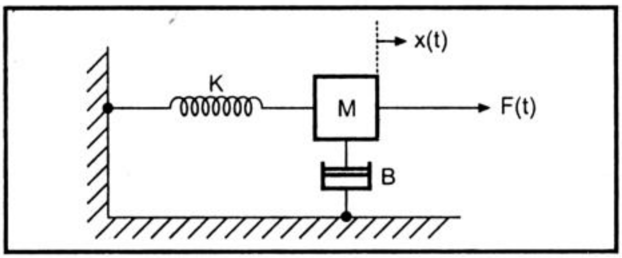
$$J\dot{\omega} + b\omega = T$$

which is a mathematical model of the mechanical rotational system considered.
The transfer function model for the system can be obtained by taking the Laplace transform of the differential equation, assuming the zero initial condition, and writing the ratio of the output (angular velocity ω) and the input (applied torque T) as follows :

$$\frac{\Omega(s)}{T(s)} = \frac{1}{Js + B}$$

where $\Omega(s) = \lambda[\omega(t)]$ and $T(s) = \lambda[T(t)]$

Consider simple mechanical system as shown in the figure below



According to Newton's Law of motion, applied force will cause displacement x(t) in spring, acceleration to mass M against frictional force having constant B.

$$\therefore \quad F(t) = Ma + Bv + kx(t)$$

where

a = acceleration

v = velocity

$$\therefore \quad F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + kx(t)$$

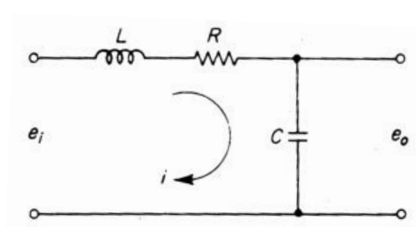
Taking Laplace,

$$F(s) = Ms^2x(s) + Bs\,x(s) + k\,x(s)$$

This is equilibrium equation for the given system.

ELECTRICAL SYSTEMS

Consider the electrical circuit shown below



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \qquad \text{....(1)}$$

$$\frac{1}{C}\int i\,dt = e_0 \qquad \text{....(2)}$$

Equations (1) and (2) give a mathematical model of the circuit.

A transfer function model of the circuit can also be obtained as follows : Taking the Laplace transforms of Equations (1) and (2), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_0(s)$$

If e_i is assumed to be the input and e_0 be the output, then the transfer function of this system is found to be

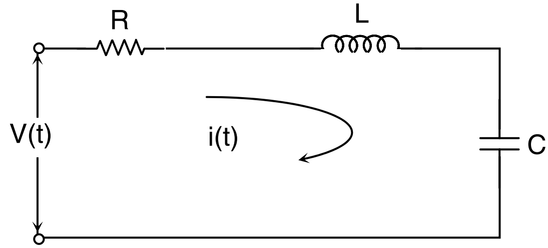
$$\frac{E_0(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

ANALOGOUE SYSTEM

Systems that can be represented by the same mathematical model but that are different physically are called analogous systems.

- The solution of the equation describing one physical system can be directly applied to analogous systems in any other field.
- Since one type of system may be easier to handle experimentally than another, instead of building and studying a mechanical system.
- In between electrical and mechanical systems there exists a fixed analogy and there exists a similarity between their equilibrium equations.
- Due to this fact it is possible to draw an electrical system which will behave exactly similar to the given mechanical system, this is called electrical analog of given mechanical system and vice versa.
- There is always an advantage to obtain electrical analog of the given mechanical system as we are well familiar with the methods of analyzing electrical network than mechanical systems.
- There are 2 methods of obtaining electrical analogous networks, namely
  1) Force – Voltage Analogy i.e. Direct Analogy
  2) Force – Current Analogy i.e. Inverse Analogy

Force Voltage Analogy (Loop Analysis) :



In this method, compared to the force in mechanical system, voltage is assumed to be analogous one.

The equation according to Kirchoff's law can be written as

$$V(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt \qquad \ldots(1)$$

Taking Laplace,

$$V(s) = I(s)R + LsI(s) + \frac{I(s)}{sC} \qquad \ldots(2)$$

Now, $i(t) = \frac{dq}{dt}$, $I(s) = s\,q(s)$ …(3)

Modifying equation (2),

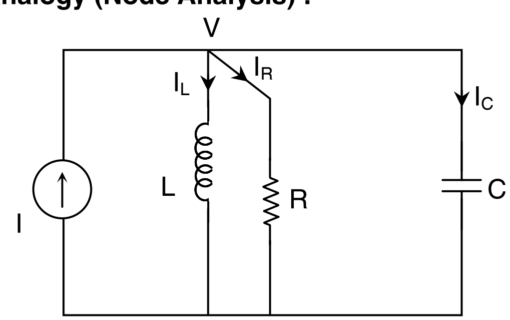
$$V(s) = Ls^2q(s) + Rsq(s) + \frac{1}{C}q(s)$$

Comparing equations F(s) and V(s) it is clear that

i) Inductance 'L' is analogous to mass M
ii) Resistance 'R' is analogous to friction B
iii) Reciprocal of capacitor i.e. 1/C is analogous to spring constant k

Translational	Rotational	Electrical
Force	Torque	Voltage
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant KNm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x}$	$\omega = \dot{\theta} = \frac{d\theta}{dt}$	Current $i = \frac{dq}{dt}$

Force Current Analogy (Node Analysis) :



In this method current is treated as analogous quantity to force in the mechanical system.
The equation according to Kirchoff's current law for above system is

$$I = I_L + I_R + I_C$$

Let node voltage be V

$$\therefore \qquad I = \frac{1}{L}\int V dt + \frac{V}{R} + C\frac{dV}{dt}$$

Taking Laplace transform,

$$I(s) = \frac{V(s)}{sL} + \frac{V(s)}{R} + sC\,V(s)$$

Now, we know that

$$V(t) = \frac{d\phi}{dt} \quad \text{where } \phi = \text{flux}$$

$$\therefore \qquad V(s) = s\phi(s)$$

Substituting in equation of I(s),

$$I(s) = Cs^2\phi(s) + \frac{1}{R}s\phi(s) + \frac{1}{L}\phi(s)$$

Comparing equations for F(s) and I(s) it is clear that,

i) Capacitor ‘C’ is analogous to mass M

ii) Reciprocal of resistance 1/R is analogous to frictional constant B

iii) Reciprocal of inductance 1/L is analogous to spring constant k.

> The elements which are in series in Force–Voltage analogy, get connected in parallel in Force–Current analogous network.

Poles

The value of s for which the system magnitude | G(s) | becomes infinity are called poles of G(s).

Zeros

The value of s for which the system magnitude | G(s) | becomes zero are called zeros of transfer function G(s).

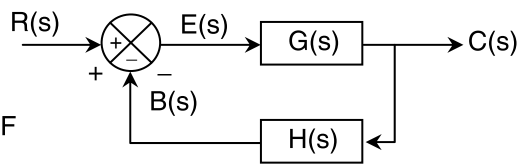
Characteristic Equation

The denominator polynomial of the transfer function of a closed loop system is called as characteristic equation and is given by

$$1 + G(s)H(s) = 0 \qquad G(s) = \frac{C(s)}{E(s)} = \text{Forward TF}$$

$$H(s) = \frac{B(s)}{C(s)} = \text{Feedback TF}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \text{closed loop TF}$$



CONCEPTS OF STATE SPACE REPRESENTATION

The analysis of multiple input – multiple output differential equation of higher order system is done using state variable analysis. The major method for analysis of feedback system is done using root locus or frequency response. But these methods are applicable to only single input–output linear time invariant system. Hence such technique leads to complications in multiple input–output time variant system. To overcome such problem state variable analysis is applied. Another limitation of the two methods is that Root locus or Frequency response provides no information regarding the internal state of the system.

> Internal state of variable is required for providing proper feedback proportional to internal variables.

Concepts of State, State Variables and State Model

A dynamic system can be represented by ordinary linear differential equation. This n–order differential equation may be expressed by first order vector matrix differential equation. This vector matrix differential equation is called a state equation..

In state variable formulation, state variables are represented by $x_1(t)$, $x_2(t)$, …… , the input by $u_1(t)$, $u_2(t)$,….. and output $y_1(t)$, $y_2(t)$, …….

$$u(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ . \\ u_m(t) \end{bmatrix}, \quad y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ . \\ y_p(t) \end{bmatrix}, \quad x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix}$$

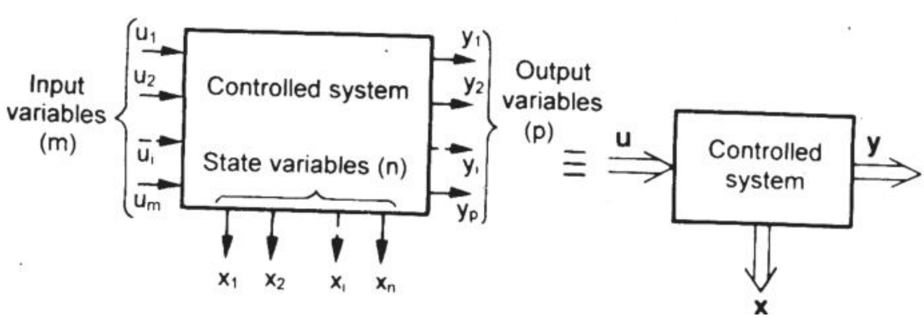


Fig. Structure of a general control system

The state variable representation can be arranged in the form of n first order differential equations.

$$\dot{x}_1 = f_j(x_1, x_2, \ldots\ldots\ldots x_n;\ u_1, u_2, \ldots\ldots\ldots u_m) \qquad y_j(t) = g_j(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_m)$$

$$\dot{x}_n = f_n(x_1, x_2, \ldots\ldots\ldots x_n;\ u_1, u_2, \ldots\ldots\ldots u_m) \qquad y_p(t) = g_p(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_m)$$

$$\dot{x}(t) = f(x(t), u(t), t) \qquad y(t) = g(x(t), u(t), t)$$

In general it can be written as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots\ldots\ldots + b_{1m}u$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + b_{21}u_1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + \ldots\ldots + \ldots\ldots\ldots\ldots\ldots + b_{nm}u_m$$

$$y_1 = c_{11}x_1 + c_{12}x_2 + \ldots\ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots\ldots\ldots + d_{1m}u_m$$

$$y_p = c_{p1}x_1 + \ldots + c_{pn}x_n + d_{p1}u_1 + \ldots + d_{pm}u_m$$

$$\therefore \quad \boxed{\dot{x}(t) = A\,x(t) + Bu(t)}$$

where

$x(t) \rightarrow$ State vector $(n \times 1)$ $\qquad$ $u(t) \rightarrow$ input vector $(m \times 1)$

$A \rightarrow$ system matrix $\qquad$ $B \rightarrow$ input matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n\times n} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & \\ \vdots & & & \\ b_{n1} & b_{n2} & & b_{nm} \end{bmatrix}_{n\times m} \quad C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \\ \vdots & & & \\ c_{p1} & c_{p2} & & c_{pn} \end{bmatrix}_{p\times n} \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & \\ \vdots & & & \\ d_{p1} & d_{p2} & & d_{pm} \end{bmatrix}_{p\times m}$$

Similarly output variable can be written as

$$y(t) = Cx(t) + Du(t)$$

where $y(t) \rightarrow$ output vector

$C \rightarrow$ output matrix

$D \rightarrow$ transmission matrix

> D is mostly not present in the state equation because there is no direct coupling between input and output.

In short : State model of linear time invariant system is,

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad \rightarrow \quad \text{state equation}$$

$$y(t) = C\,x(t) + D\,u(t) \quad \rightarrow \quad \text{output equation}$$

State variable model is not unique, but number of elements in state vector is equal and minimum.

Note :

In time varying systems, the coefficients A, B, C and D are not constant but have time dependency factor.

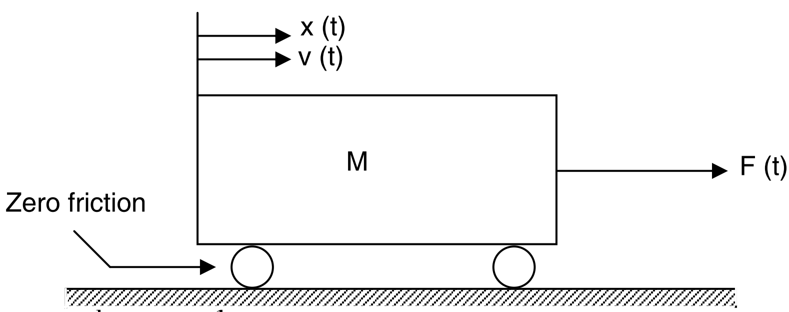
State variable analysis can be used to solve both linear and nonlinear systems. The state of a system is nothing but status of the system.

A system uses three types of variables to represent the dynamics of the system.
a) input variables
b) output variables
c) state variables

The state of a dynamic system is a minimal set of variables (i.e. state variables) such that knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behavior of the system for $t \geq t_o$.

A mathematical model used to represent dynamics of a system utilizes three types of Variables called the input, output and the state variable.

Consider the following mechanical system, where mass M is acted upon by the force F(t). The system can be characterized as,



$$\frac{d}{dt} v(t) = \frac{1}{M} F(t)$$

$$\frac{d}{dt} x(t) = v(t)$$

The state of the system of figure at any time t is given by the variables x (t) & v (t) which are called state variables of the system.

For time–varying systems, the function f is dependent on time as well and the vector equation may be written as

$$\dot{X}(t) = f(x(t), u(t), t) \quad \ldots (1)$$

For time–invariant system,

The ‘n’ differential equations may be written in vector notation as,

$$\dot{X}(t) = f(x(t), u(t)) \qquad \dots \quad (2)$$

The equations (1) & (2) are the state equations for time–varying & time–invariant systems respectively. The state vector x determines a point (called a state point) in an n–dimensional space, called state space.

The output y (t) can in general be expressed in terms of the state x (t) & input u (t) as,

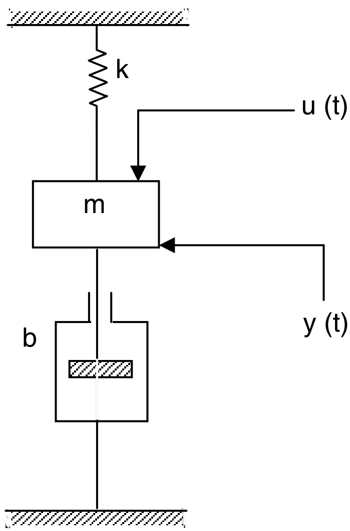
$y(t) = g(x(t), u(t))$; time – invariant systems

$y(t) = g(x(t), u(t), t)$; time – varying systems

The above equations are the output equations for time – invariant & time varying systems respectively.

Applying State Space Representation

Consider the mechanical system shown in the figure below. Assume the system is linear & the external force u (t) is the input to the system. The displacement y (t) of the mass is the output. The displacement y (t) is measured from the equilibrium position in the absence of the external force. This is a single–input, single–output system.



From this diagram, the system equation is,

$$m\ddot{y} + b\dot{y} + ky = u$$

This system is of second order.

This means that the system involves two integrations. Let us consider two state variables x_1 (t) and x_2 (t) as,

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t)$

Then we obtain,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(-ky - b\dot{y}\right) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is,

$y = x_1$

In a vector matrix form, the above equations can be written as,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u \qquad \ldots \text{ (1)}$$

The output equation may be written as,

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad \ldots \text{ (2)}$$

Equation (1) is the state equation & equation (2) is the output equation for the system. These equations in standard form can be written as,

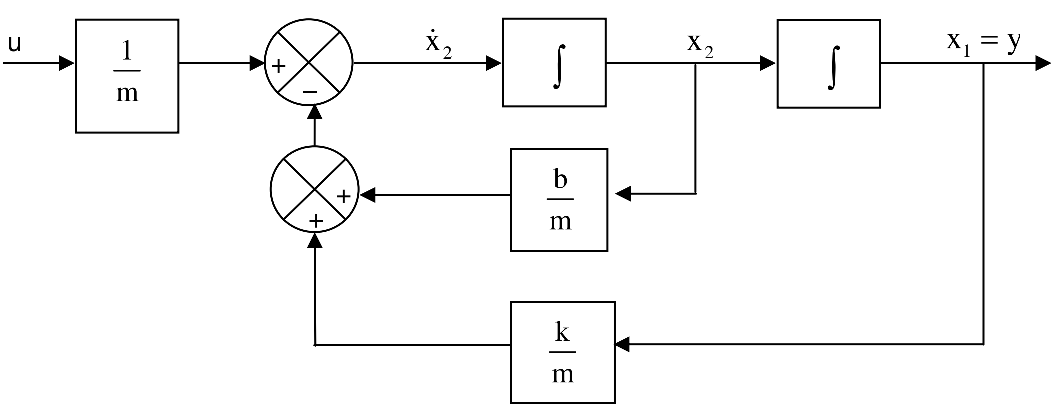
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where, $A = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}$, $\qquad B = \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$

$C = \begin{bmatrix}1 & 0\end{bmatrix}$, $\qquad D = 0$

The figure below shows the block diagram for the above system where the outputs of the integrators are state variables.



Correlation between transfer functions & state – space equations

To derive the transfer function of a single – input, single – output system from the state – space equations, the following method should be followed.

Consider a system whose transfer function is given by,

$$\frac{Y(s)}{U(s)} = G(s) \qquad \ldots \quad (a)$$

This system may be represented in state space by following equations –

$$\dot{X} = Ax + Bu \qquad \ldots \quad (b)$$

$$\dot{Y} = Cx + Du \qquad \ldots \quad (c)$$

where x$\rightarrow$ state vector
u$\rightarrow$ input & y$\rightarrow$ output

The Laplace Transforms of equation (b) & (c) are,

$$sX(s) - X(0) = AX(s) + BU(s) \qquad \ldots \quad (d)$$

$$Y(s) = CX(s) + DU(s) \qquad \ldots \quad (e)$$

As we know that, transfer function is defined as laplace transform of output to laplace transform of input when initial conditions are zero.

Thus we assume X (0) =0 in equation (d).

sX (s) – AX (s) = BU (s)
or (sI – A) X(s) = BU (s)

Pre–multiplying by $(sI - A)^{-1}$ to both sides,

$x(s) = Bu(s)\,(sI - A)^{-1}$

Substituting the above equation in (e),

$y(s) = C(sI - A)^{-1}\,Bu(s) + Du(s)$
$y(s) = [C\,(sI - A)^{-1}\,B + D]\,u(s)$

$$\therefore \frac{Y(s)}{U(s)} = C\left(sI - A\right)^{-1} B + D$$

Comparing the above equation with equation (a),
$G(s) = C\,(sI - A)^{-1}\,B + D$

This is the transfer – function expression in terms of A, B, C and D.
The right hand side of above equation involves $(sI - A)^{-1}$. Hence G (s) can be written as,

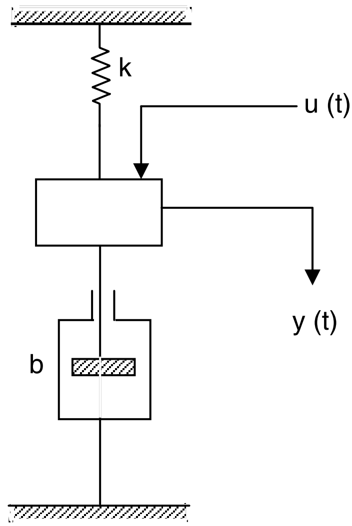
$$G(s) = \frac{Q(s)}{|sI - A|}$$

where

Q (s) is a polynomial in s. Thus, $|sI - A|$ is equal to the characteristic polynomial of G (s).

Or we can say that, eigen values of A are identical to the poles of G (s).

Example :



Consider the mechanical system shown in figure below, The state space equations for the system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

and $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Obtain the transfer function for the system from the state space equations.

Solution :

By substituting A, B, C & D into equation,

$$G(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Since $\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \dfrac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$

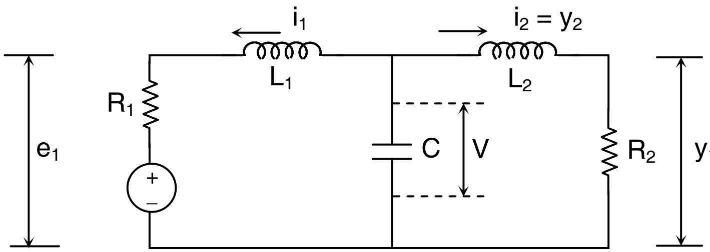
We get,

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms^2 + bs + k}$$

This is the transfer function of the system.

STATE SPACE REPRESENTATION USING PHYSICAL VARIABLES

Consider a simple electrical system of RLC network as shown in the figure.



For analysis of this circuit we can adopt normal law of electrical network, but in this case we are not familiar with the initial conditions of the elements. These we switch over to state variable analysis because of minimum available information.

Consider state variables as _____

$$x_1(t) = v(t), \quad x_2(t) = i_1(t), \quad x_3(t) = i_2(t)$$

The differential equations of the RLC network are,

$$i_1 + i_2 + c\frac{dv}{dt} = 0$$

$$L_1\frac{di_1}{dt} + R_1 i_1 + e = v$$

$$L_2\frac{di_2}{dt} + R_2 i_2 = v$$

Rearranging the terms we get,

$$\frac{dv}{dt} = -\frac{1}{C}i_1 - \frac{1}{C}i_2 \quad = \dot{x}_1$$

$$\frac{di_1}{dt} = \frac{1}{L_1}v - \frac{R_1}{L_1}i_1 - \frac{1}{L_1}e \quad = \dot{x}_2$$

$$\frac{di_2}{dt} = \frac{1}{L_2}v - \frac{R_2}{L_2}i_2 \quad = \dot{x}_3$$

Now we can represent the above equations in the form state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} u$$

Similarly, assume voltage across and current through R_2 are output variables y_1 and y_2 respectively.

$$\therefore \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State variable model analysis can be extended to electromechanical system also.

CONTROLLABILITY AND OBSERVABILITY

Controllability

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_f)$ in specified time by a control vector (say u(t)).

Test for controllability

For an nth order system to be controllable, rank of the matrix $[B, AB, \ldots, A^{n-1} B]_{1 \times n}$ should be n.

Example :

Consider a second order linear system.

$$\dot{x}_1(t) = -2x_1(t) + 5x_2(t)$$
$$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$$

Solution

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Here, $A = \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$[B \cdot AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} : \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$|B : AB| = \begin{vmatrix} 0 & 5 \\ 1 & -1 \end{vmatrix}$$

$\therefore \quad |B : AB| = -5$

$\therefore \quad |B : AB| \neq 0$

$\therefore \quad$ rank = 2 and order = 2

$\boxed{\therefore \quad \text{System is controllable.}}$

> If rank ≠ order, system is uncontrollable.

Observability

A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the output y(t) over a finite time interval.

Test for observability

For observability, for 2^{nd} order system,

$$\text{Rank of } \begin{vmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{vmatrix}_{np\times n} \quad \text{be n}$$

Example :

Consider the following example :

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Solution :

Here, $A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$C = [1 \quad 2]$$

$$CA = [1 \quad 2] \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= [-1 \quad -8]$$

$$\begin{vmatrix} C \\ CA \end{vmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -8 \end{bmatrix}$$

$$= -6$$

$$\neq 0 \qquad \qquad ....(1)$$

$$AC^T = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$$

$$|C^T : AC^T| = \begin{vmatrix} 1 & -1 \\ 2 & -8 \end{vmatrix}$$

$$= -6$$

$$\neq 0 \qquad \qquad ....(2)$$

$\boxed{\text{From (1) and (2), system is observable.}}$

Test for Stability

For stability analysis, solve | sI – A| = 0 and get the characteristic equation.

Using Routh's array, find the stability of the system.

Consider the following example :

Let $A = \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix}$

$$|sI - A| = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (s+2) & -5 \\ (-2) & (s+1) \end{vmatrix} = 0$$

$$s^2 + 3s + 2 - 10 = 0$$

$$s^2 + 3s - 8 = 0$$

Using Routh's array,

$$\begin{array}{c|cc} s^2 & 1 & -8 \\ s^1 & 3 & \\ s^0 & -8 & \end{array}$$

Since there is one sign change in the 1st row, the system is unstable.

LIST OF FORMULAE

- The transfer function of simple closed loop system with negative feedback is

  $$\text{T.F.} = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\ \ H(s)}$$

  where G(s) is the forward path gain
  H(s) is the feedback gain

- For gear system:

  $$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

- Equivalent Mechanical Systems

  For mass → $M\frac{d^2x}{dt^2}$

  For friction → $B\frac{dx}{dt}$

  For spring → kx

- In mechanical systems, the force can be expressed as

$$F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + kx(t)$$

where $x(t)$ is the displacement
M is the mass
B is the frictional constant
k is the spring constant

Translational	Rotational	Electrical
Force	Torque	Voltage
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant kNm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x}$	$\dot{\theta} = \frac{dq}{dt}$	Current $i = \frac{dq}{dt}$

LMR (LAST MINUTE REVISION)

- The control systems are classified into two types :
  - Open loop control system
    Here, the control action is totally independent of the output.
  - Closed loop control system
    In this case, the controlling action is some how dependent on the output.

- There are two types of feedbacks.
  - Positive Feedback
  - Negative Feedback

- When feedback is given the error between system input and output is reduced. However improvement of error is not only advantage. The effects of feedback are
  - Gain is reduced by a factor $\frac{G}{1 \pm GH}$.
  - Reduction of parameter variation by a factor $1 \pm GH$.
  - Improvement in sensitivity.
  - Stability may be affected.

- **Test of Controllability**

  A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_f)$ in specified time by a control vector (say u(t)).

Test for Controllability

For an n^{th} order system to be controllable, rank of the matrix $[B:AB, \ldots\ldots A^{n-1}B]$ should be n.

- **Test of Observability**

A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the output y(t) over a finite time interval.

Test for observability

For 2^{nd} order system, Rank of $\begin{vmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{vmatrix}_{np\times n}$ be n .

- **Test for Stability**

For stability analysis, solve $|SI - A| = 0$ and get the characteristic equation.

Using Routh's array, we can find the stability of the system.

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